Exercise 5

Consider the equation
\[ u_x + yu_y = 0 \]
with the boundary condition \( u(x, 0) = \phi(x) \).

(a) For \( \phi(x) \equiv x \), show that no solution exists.

(b) For \( \phi(x) \equiv 1 \), show that there are many solutions.

Solution

On the paths defined by
\[ \frac{dy}{dx} = y, \]  \hspace{1cm} (1)
the PDE reduces to an ODE,
\[ \frac{du}{dx} = 0. \]  \hspace{1cm} (2)
That is, \( u = u(x, y) \) is constant on the characteristics defined by (1). Integrating (2), we find that
\[ u(x, \xi) = f(\xi), \]
where \( f \) is an arbitrary function of the characteristic coordinate, \( \xi \). Solving (1) by separation of variables gives
\[ \frac{dy}{y} = dx \]
\[ \ln |y| = x + C \]
\[ |y| = e^{x+C} \]
\[ y = \pm e^C e^x \]
\[ y = \xi e^x. \]
Solving for \( \xi \) gives
\[ \xi = ye^{-x}. \]
Therefore,
\[ u(x, y) = f (ye^{-x}) . \]
We can check that this is the solution of the PDE.
\[ u_x = -ye^{-x} f' \]
\[ u_y = e^{-x} f' \]
\[ u_x + yu_y = 0, \]
so this is the correct solution. Shown below in Figure 1 are the characteristic curves in the \( xy \)-plane for various values of \( \xi \) along with the line \( y = 0 \) (where the boundary condition is defined). Note that because the data curve, \( y = 0 \), only intersects the \( \xi = 0 \) characteristic, the solution is only defined for \( y = 0 \) and all \( x \).
Figure 1: Plot of the characteristic curves and the data curve for $-5 < x < 5$ and $-5 < y < 5$.

Part (a)

The boundary condition is $\phi(x) = x$ when $y = 0$, so

$$u(x, 0) = f(0) = x.$$ 

$u = x$ doesn’t satisfy the PDE. Unfortunately, a solution cannot be determined from this boundary condition.

Part (b)

The boundary condition is $\phi(x) = 1$ when $y = 0$, so

$$u(x, 0) = f(0) = 1.$$ 

$u = 1$ does satisfy the PDE. From this, all we can say about $u$ is that

$$u(x, y) = \begin{cases} 
1 & \text{when } y = 0 \\
 f \left( ye^{-x} \right) & \text{all other } x \text{ and } y.
\end{cases}$$

Therefore, the solution is not unique.