Exercise 6

Solve the equation $u_x + 2xy^2 u_y = 0$.

Solution

On the paths defined by

$$\frac{dy}{dx} = 2xy^2, \tag{1}$$

the PDE reduces to an ODE,

$$\frac{du}{dx} = 0. \tag{2}$$

That is, $u = u(x, y)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$u(x, \xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by separation of variables gives

$$\begin{aligned}
\frac{dy}{y^2} &= 2x\,dx \\
-\frac{1}{y} &= x^2 + C \\
-C &= x^2 + \frac{1}{y} \\
\xi &= x^2 + \frac{1}{y}.
\end{aligned}$$

Therefore,

$$u(x, y) = f\left(x^2 + \frac{1}{y}\right).$$

We can check that this is the solution of the PDE.

$$\begin{aligned}
u_x &= 2xf' \\
u_y &= -\frac{1}{y^2}f'
\end{aligned}$$

$u_x + 2xy^2 u_y = 0$, so this is the correct solution. Shown below in Figure 1 are the characteristic curves in the xy-plane for various values of ξ.

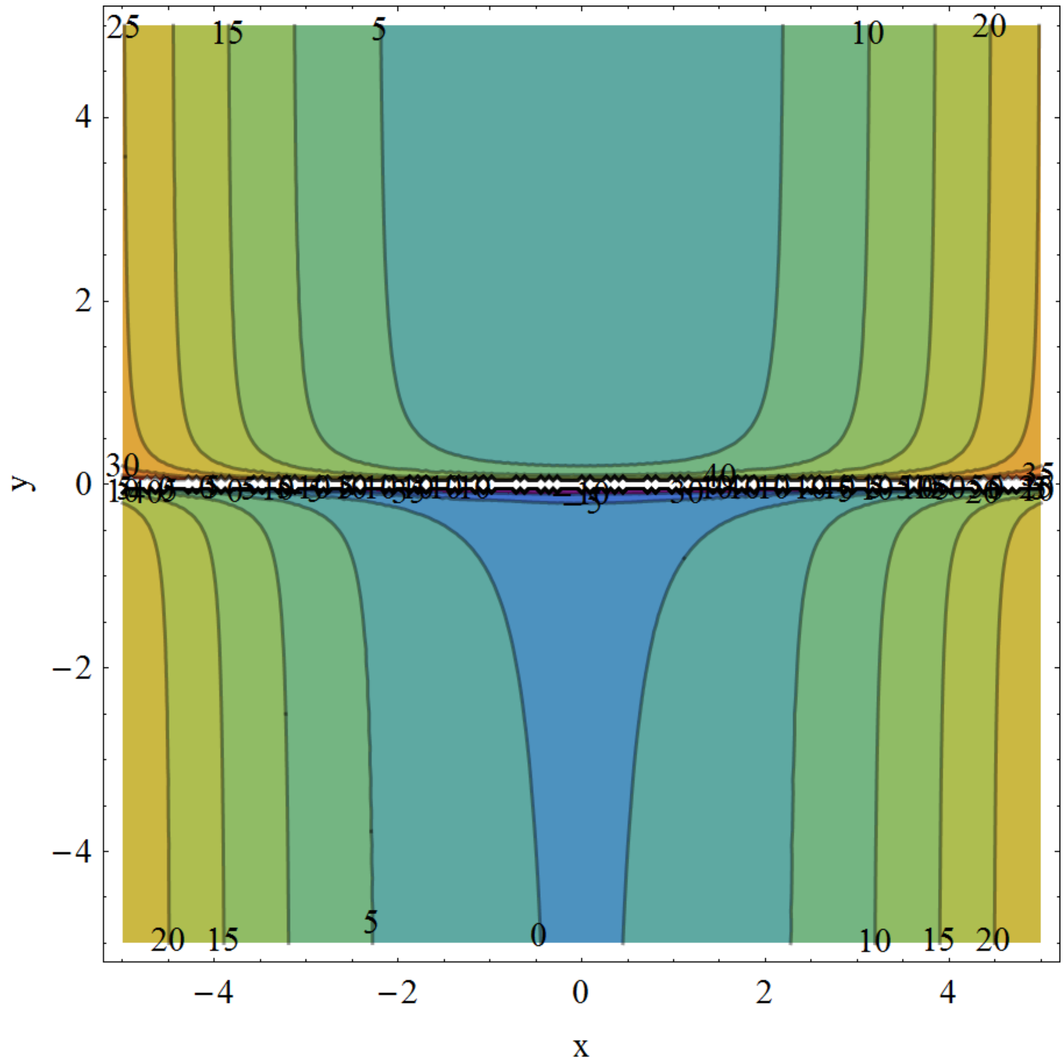

Figure 1: Plot of the characteristic curves for $-5 < x < 5$ and $-5 < y < 5$.