Exercise 1

What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

Solution

The general form of a second-order PDE is

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$, is equal to, greater than, or less than 0, respectively. In other words,

$$\text{the PDE is} \begin{cases} \text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\ \text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\ \text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0 \end{cases}.$$

Part (a)

The given equation can be simplified since $u_{xy} = u_{yx}$.

$$u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0$$

The relevant coefficients are $a_{11} = 1$, $a_{12} = -2$, and $a_{22} = 1$. So

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = (-2)^2 - 1 \cdot 1 = 3.$$

Therefore, the PDE is hyperbolic.

Part (b)

The relevant coefficients are $a_{11} = 9$, $a_{12} = 3$, and $a_{22} = 1$. So

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = 3^2 - 9 \cdot 1 = 0.$$

Therefore, the PDE is parabolic.