# Exercise 6

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. (*Hint:* Substitute $v = u_y$.)

(c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?

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**Solution**

## Part (a)

The general form of a second-order PDE is

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.$$

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, $\mathscr{D} = a_{12}^2 - a_{11}a_{22}$, is equal to, greater than, or less than 0, respectively. In other words,

$$\text{the PDE is} \begin{cases} \text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\ \text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\ \text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0 \end{cases}.$$

For this equation the relevant coefficients are $a_{11} = 0$, $a_{12} = 1/2$, and $a_{22} = 0$. Hence, the discriminant is

$$\mathscr{D} = a_{12}^2 - a_{11}a_{22} = \left(\frac{1}{2}\right)^2 - 0 \cdot 0 = \frac{1}{4}.$$

Therefore, the PDE is hyperbolic.

## Part (b)

Following the hint, substitute $v = u_y$ into the PDE to get

$$3v + v_x = 0$$

Because this equation is a first order linear differential equation, it can be solved by multiplying both sides by an integrating factor,

$$I(x) = e^{\int^x 3\,ds} = e^{3x}.$$

$$e^{3x}v_x + 3e^{3x}v = 0$$

$$\left(e^{3x}v\right)_x = 0$$

Integrate both sides partially with respect to $x$.

$$e^{3x}v = f(y),$$

where $f$ is an arbitrary function of $y$.

$$v = e^{-3x} f(y)$$
$$u_y = e^{-3x} f(y)$$

Solve for $u$ by integrating partially with respect to $y$.

$$u(x, y) = e^{-3x} g(y) + h(x),$$

where $g$ is an arbitrary function of $y$ and $h$ is an arbitrary function of $x$. We can verify that this is the general solution of the PDE.

$$u_y = e^{-3x} g'$$
$$u_{yx} = -3e^{-3x} g'$$

Since $u_{yx} = u_{xy}$, $3u_y + u_{xy} = 0$, so this is the correct solution for $u(x, y)$.

**Part (c)**

The boundary conditions yield the following equations:

$$u(x, 0) = e^{-3x} \quad \rightarrow \quad e^{-3x} g(0) + h(x) = e^{-3x}$$
$$u_y(x, 0) = 0 \quad \rightarrow \quad e^{-3x} g'(0) = 0.$$

These equations tell us that $h(x)$ must evaluate to 0, $g(0)$ must evaluate to 1, and that $g'(0)$ must evaluate to 0. Thus, $u(x, y) = e^{-3x}$ is a solution of the boundary value problem and that $u(x, y) = e^{-3x} e^{-y^2}$ is a solution and that $u(x, y) = e^{-3x} e^{y^4}$ is a solution, etc. In general, any solution of the form,

$$u(x, y) = e^{-3x} e^{F(y)}, \quad \text{where } F(y) \text{ is arbitrary, provided that } F(0) = F'(0) = 0,$$

satisfies the PDE and the boundary conditions. Therefore, the solution is not unique.