Exercise 1

What is the type of each of the following equations?

(a) \( u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0 \).

(b) \( 9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0 \).

Solution

The general form of a second-order PDE is

\[
a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.
\]

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, \( \mathcal{D} = a_{12}^2 - a_{11}a_{22} \), is equal to, greater than, or less than 0, respectively. In other words,

\[
\begin{align*}
\text{the PDE is} & \quad \begin{cases} 
\text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\
\text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\
\text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0
\end{cases}
\end{align*}
\]

Part (a)

The given equation can be simplified since \( u_{xy} = u_{yx} \).

\[
u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0
\]

The relevant coefficients are \( a_{11} = 1, a_{12} = -2, \) and \( a_{22} = 1 \). So

\[
\mathcal{D} = a_{12}^2 - a_{11}a_{22} = (-2)^2 - 1 \cdot 1 = 3.
\]

Therefore, the PDE is hyperbolic.

Part (b)

The relevant coefficients are \( a_{11} = 9, a_{12} = 3, \) and \( a_{22} = 1 \). So

\[
\mathcal{D} = a_{12}^2 - a_{11}a_{22} = 3^2 - 9 \cdot 1 = 0.
\]

Therefore, the PDE is parabolic.