Exercise 3

Among all the equations of the form (1), show that the only ones that are unchanged under all rotations (*rotationally invariant*) have the form $a(u_{xx} + u_{yy}) + bu = 0$.

Solution

The general form of a second-order PDE is given by (1) in the book:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0. \tag{1}$$

Suppose we define a new pair of axes in the xy-plane by making a rotation of angle α. We can

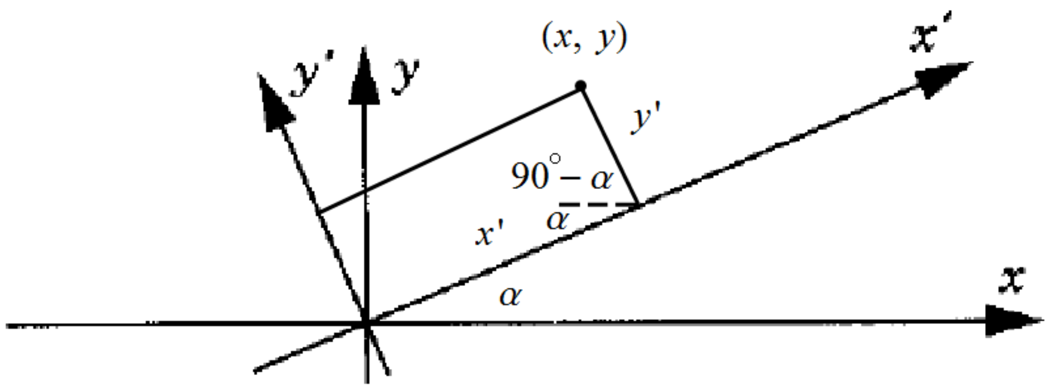

Figure 1: The new x' and y' axes are made from rotating the x and y axes by angle α.

write x and y in terms of x' and y' to get a transformation.

$$x = x' \cos\alpha - y' \cos(90° - \alpha)$$
$$y = x' \sin\alpha + y' \sin(90° - \alpha)$$

Simplifying this, we get

$$x = x' \cos\alpha - y' \sin\alpha$$
$$y = x' \sin\alpha + y' \cos\alpha$$

Solving for the new variables, x' and y', gives us

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha.$$

Thus, if there's a rotation in the plane by angle α, the transformation above gives us the new variables in terms of the old ones. Now we want to make this change of variables in (1). We will have to use the chain rule to find expressions in terms of the new variables.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u_x}{\partial y'}\frac{\partial y'}{\partial x} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial u_y}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u_y}{\partial y'}\frac{\partial y'}{\partial y}
\end{aligned}$$

We get the following.

$$\begin{aligned}
u_x &= u_{x'}\cos\alpha - u_{y'}\sin\alpha \\
u_y &= u_{x'}\sin\alpha + u_{y'}\cos\alpha \\
u_{xx} &= (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\cos\alpha - (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{y'}\sin\alpha \\
u_{yy} &= (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha
\end{aligned}$$

Remarkably, if we add u_{xx} and u_{yy} together, it turns out to be the same as the sum of $u_{x'x'}$ and $u_{y'y'}$. (Note that $u_{x'y'} = u_{y'x'}$.)

$$\begin{aligned}
u_{xx} + u_{yy} &= (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) + u_{x'y'}(-\cancel{2\sin\alpha\cos\alpha} + \cancel{2\sin\alpha\cos\alpha}) \\
u_{xx} + u_{yy} &= u_{x'x'} + u_{y'y'}
\end{aligned}$$

Therefore, if $a_{11} = a_{22} = a$ and $a_{12} = a_1 = a_2 = 0$ and $a_0 = b$ in (1), the resulting equation, $a(u_{xx} + u_{yy}) + bu = 0$, is rotationally invariant.