Exercise 4

What is the type of the equation
\[ u_{xx} - 4u_{xy} + 4u_{yy} = 0? \]

Show by direct substitution that \( u(x, y) = f(y + 2x) + xg(y + 2x) \) is a solution for arbitrary functions \( f \) and \( g \).

Solution

The general form of a second-order PDE is
\[
a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0.
\]

It is classified as parabolic, hyperbolic, or elliptic depending on whether the discriminant, \( \mathcal{D} = a_{12}^2 - a_{11}a_{22} \), is equal to, greater than, or less than 0, respectively. In other words,
\[
\begin{align*}
\text{the PDE is} & \quad \begin{cases}
\text{hyperbolic} & \text{if } a_{12}^2 - a_{11}a_{22} > 0 \\
\text{parabolic} & \text{if } a_{12}^2 - a_{11}a_{22} = 0 \\
\text{elliptic} & \text{if } a_{12}^2 - a_{11}a_{22} < 0
\end{cases}
\end{align*}
\]

The relevant coefficients are \( a_{11} = 1 \), \( a_{12} = -2 \), and \( a_{22} = 4 \). So
\[ \mathcal{D} = a_{12}^2 - a_{11}a_{22} = (-2)^2 - 1 \cdot 4 = 0. \]

Therefore, the PDE is parabolic. We can verify that
\[ u(x, y) = f(y + 2x) + xg(y + 2x) \]
is the correct solution.

\[
\begin{align*}
u_x & = 2f' + g + 2xg' \\
u_{xx} & = 2f'' + 2 + 2g' + 2g'' + 4xy'' \\
u_{xy} & = 2f'' + g' + 2xy'' \\
u_y & = f' + xg' \\
u_{yy} & = f'' + xg''
\end{align*}
\]
So
\[
\begin{align*}
u_{xx} - 4u_{xy} + 4u_{yy} & = 4f'' + 4g' + 2xy'' - 8f'' - 8g' - 8xy'' + 4f'' + 4g'' \\
u_{xx} - 4u_{xy} + 4u_{yy} & = 0.
\end{align*}
\]

Hence, \( u(x, y) \) is the correct solution.