Exercise 5

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$ by a change of dependent variable $u = ve^{\alpha x + \beta y}$ and then a change of scale $y' = \gamma y$.

Solution

Making the change of variables, $u = ve^{\alpha x + \beta y}$, we get the following derivatives.

$$\begin{aligned}
u_x &= v_x e^{\alpha x + \beta y} + \alpha v e^{\alpha x + \beta y} \\
u_y &= v_y e^{\alpha x + \beta y} + \beta v e^{\alpha x + \beta y} \\
u_{xx} &= v_{xx} e^{\alpha x + \beta y} + \alpha v_x e^{\alpha x + \beta y} + \alpha v_x e^{\alpha x + \beta y} + \alpha^2 v e^{\alpha x + \beta y} \\
u_{xx} &= (v_{xx} + 2\alpha v_x + \alpha^2 v) e^{\alpha x + \beta y} \\
u_{yy} &= v_{yy} e^{\alpha x + \beta y} + \beta v_y e^{\alpha x + \beta y} + \beta v_y e^{\alpha x + \beta y} + \beta^2 v e^{\alpha x + \beta y} \\
u_{yy} &= (v_{yy} + 2\beta v_y + \beta^2 v) e^{\alpha x + \beta y}
\end{aligned}$$

Plugging these expressions into the PDE gives us

$$(v_{xx} + 2\alpha v_x + \alpha^2 v) e^{\alpha x + \beta y} + 3(v_{yy} + 2\beta v_y + \beta^2 v) e^{\alpha x + \beta y} - 2(v_x + \alpha v) e^{\alpha x + \beta y} \\ + 24(v_y + \beta v) e^{\alpha x + \beta y} + 5ve^{\alpha x + \beta y} = 0.$$

Simplifying this yields

$$e^{\alpha x + \beta y} \left[v_{xx} + 3v_{yy} + 2v_x(\alpha - 1) + 6v_y(\beta + 4) + v(\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5) \right] = 0.$$

Now divide both sides by $e^{\alpha x + \beta y}$ and make the second change of variables, $y' = \gamma y$. By the chain rule, $v_y = v_{y'} \gamma$ and $v_{yy} = v_{y'y'} \gamma^2$. Thus, the equation becomes

$$v_{xx} + 3\gamma^2 v_{y'y'} + 2v_x(\alpha - 1) + 6\gamma v_{y'}(\beta + 4) + v(\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5) = 0.$$

If we let $\alpha = 1$ and $\beta = -4$ and $\gamma^2 = 1/3$, we obtain

$$v_{xx} + v_{y'y'} - 44v = 0,$$

and this is of the desired form, $v_{xx} + v_{yy} + cv = 0$.