Exercise 2

Solve the wave equation in the rectangle $R = \{0 < x < a,\ 0 < y < b\}$, with homogeneous Dirichlet conditions on the boundary, and the initial conditions $u(x, y, 0) = xy(b - y)(a - x)$, $u_t(x, y, 0) \equiv 0$.

Solution

The initial boundary value problem to solve is

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u, \quad 0 < x < a,\ 0 < y < b,\ t > 0 \\ u(0, y, t) &= 0 \qquad u(x, 0, t) = 0 \\ u(a, y, t) &= 0 \qquad u(x, b, t) = 0 \\ u(x, y, 0) &= xy(b - y)(a - x) \\ u_t(x, y, 0) &= 0. \end{aligned}$$

Because the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve for u. Assume a product solution of the form $u(x, y, t) = X(x)Y(y)T(t)$ and substitute it into the PDE

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad \rightarrow \quad XYT'' = c^2(X''YT + XY''T)$$

and the boundary conditions.

$$\begin{aligned} u(0, y, t) &= 0 & &\rightarrow & X(0)Y(y)T(t) &= 0 & &\rightarrow & X(0) &= 0 \\ u(a, y, t) &= 0 & &\rightarrow & X(a)Y(y)T(t) &= 0 & &\rightarrow & X(a) &= 0 \\ u(x, 0, t) &= 0 & &\rightarrow & X(x)Y(0)T(t) &= 0 & &\rightarrow & Y(0) &= 0 \\ u(x, b, t) &= 0 & &\rightarrow & X(x)Y(b)T(t) &= 0 & &\rightarrow & Y(b) &= 0 \end{aligned}$$

Separate variables in the PDE: divide both sides by c^2XYT so that the left side is a function of t and the right side is a function of x and y.

$$\underbrace{\frac{T''}{c^2T}}_{\text{function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{function of } x \text{ and } y}$$

The only way a function of t can be equal to a function of x and y is if both are equal to a constant λ.

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Subtract both sides of the second equation by Y''/Y.

$$\underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to another constant μ.

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu$$

As a result of applying the method of separation of variables, the PDE has been reduced to a system of three ODEs—one in x, one in y, and one in t.

$$\left.\begin{aligned} \frac{T''}{c^2T} &= \lambda \\ \frac{X''}{X} &= \mu \\ \lambda - \frac{Y''}{Y} &= \mu \end{aligned}\right\}$$

Start by solving the ODE for X.

$$X'' = \mu X$$

Suppose first that μ is positive: $\mu = \alpha^2$.

$$X'' = \alpha^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions for X to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(a) &= C_1 \cosh \alpha a + C_2 \sinh \alpha a = 0 \end{aligned}$$

The second equation reduces to $C_2 \sinh \alpha a = 0$. No nonzero value of α satisfies this equation, so $C_2 = 0$. The trivial solution $X(x) = 0$ is obtained, which means there are no positive eigenvalues. Suppose now that μ is zero: $\mu = 0$. The ODE for X becomes

$$X'' = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$X(x) = C_3 x + C_4$$

Apply the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(a) &= C_3 a + C_4 = 0 \end{aligned}$$

The second equation reduces to $C_3 a = 0$, so $C_3 = 0$. The trivial solution $X(x) = 0$ is obtained, which means zero is not an eigenvalue. Suppose now that μ is negative: $\mu = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(a) &= C_5 \cos \beta a + C_6 \sin \beta a = 0 \end{aligned}$$

The second equation reduces to $C_6 \sin \beta a = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\begin{aligned} \sin \beta a &= 0 \\ \beta a &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{a}. \end{aligned}$$

There are negative eigenvalues $\mu = -n^2\pi^2/a^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_6 \sin \beta x \quad \to \quad X_n(x) = \sin \frac{n\pi x}{a}. \end{aligned}$$

With $\mu = -n^2\pi^2/a^2$, the ODE for Y becomes

$$\begin{aligned} \lambda - \frac{Y''}{Y} &= -\frac{n^2\pi^2}{a^2} \\ Y'' &= -\left(-\lambda - \frac{n^2\pi^2}{a^2}\right) Y. \end{aligned}$$

Suppose first that λ is positive: $\lambda = \xi^2$.

$$Y'' = \left(\xi^2 + \frac{n^2\pi^2}{a^2}\right) Y.$$

The general solution is written in terms of hyperbolic sine and cosine.

$$Y(y) = C_7 \cosh \sqrt{\xi^2 + \frac{n^2\pi^2}{a^2}}\, y + C_8 \sinh \sqrt{\xi^2 + \frac{n^2\pi^2}{a^2}}\, y$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} Y(0) &= C_7 = 0 \\ Y(b) &= C_7 \cosh \sqrt{\xi^2 + \frac{n^2\pi^2}{a^2}}\, b + C_8 \sinh \sqrt{\xi^2 + \frac{n^2\pi^2}{a^2}}\, b = 0 \end{aligned}$$

The second equation reduces to $C_8 \sinh \sqrt{\xi^2 + \frac{n^2\pi^2}{a^2}}\, b = 0$. No nonzero value of ξ satisfies this equation, so C_8 must be zero. The trivial solution $Y(y) = 0$ is obtained, so there are no positive eigenvalues for λ. Suppose secondly that λ is zero: $\lambda = 0$.

$$Y'' = \frac{n^2\pi^2}{a^2} Y$$

Like before, the general solution is written in terms of hyperbolic sine and hyperbolic cosine, which leads to the trivial solution. Zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\zeta^2$. The ODE for Y becomes

$$Y'' = -\left(\zeta^2 - \frac{n^2\pi^2}{a^2}\right) Y.$$

The general solution is written in terms of sine and cosine.

$$Y(y) = C_9 \cos\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}y + C_{10}\sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}y$$

Apply the boundary conditions to determine C_9 and C_{10}.

$$\begin{aligned} Y(0) &= C_9 = 0 \\ Y(b) &= C_9 \cos\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}b + C_{10}\sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}b = 0 \end{aligned}$$

The second equation reduces to $C_{10}\sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}b = 0$. To avoid getting the trivial solution, we insist that $C_{10} \neq 0$. Then

$$\begin{aligned} \sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}b &= 0 \\ \sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}b &= m\pi, \quad m = 1, 2, \ldots \\ \sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}} &= \frac{m\pi}{b} \\ \zeta^2 - \frac{n^2\pi^2}{a^2} &= \frac{m^2\pi^2}{b^2} \\ \zeta^2 &= \frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{a^2} \\ \zeta_{mn} &= \pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}. \end{aligned}$$

There are negative eigenvalues $\lambda = -\pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_9 \cos\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}y + C_{10}\sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}y \\ &= C_{10}\sin\sqrt{\zeta^2 - \frac{n^2\pi^2}{a^2}}y \quad \to \quad Y_m(y) = \sin\frac{m\pi y}{b}. \end{aligned}$$

With $\lambda = -\pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$, the ODE for T becomes

$$T'' = -c^2\pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)T.$$

The general solution is written in terms of sine and cosine.

$$T(t) = C_{11}\cos c\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}t + C_{12}\sin c\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}t$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)T(t)$ over all the eigenvalues.

$$u(x, y, t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{a}\left[\sum_{m=1}^{\infty}\left(A_{mn}\cos c\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}t + B_{mn}\sin c\pi\sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}}t\right)\sin\frac{m\pi y}{b}\right]$$

Simplify the right side.

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}\cos c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}t+B_{mn}\sin c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}t\right)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

The final task is to use the initial conditions, $u(x,y,0)=xy(b-y)(a-x)$ and $u_t(x,y,0)=0$, to determine the coefficients. Take a derivative of u with respect to t.

$$u_t(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}\left(-A_{mn}\sin c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}t+B_{mn}\cos c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}t\right)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

Apply the initial conditions.

$$\begin{aligned}
u(x,y,0)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}=xy(b-y)(a-x)\\
u_t(x,y,0)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}B_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}=0
\end{aligned}$$

By inspection we see that $B_{mn}=0$. To find A_{mn}, multiply both sides of the first equation by $\sin(p\pi y/b)$, where p is an integer,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi y}{b}=xy(b-y)(a-x)\sin\frac{p\pi y}{b}$$

and then integrate both sides with respect to y from 0 to b.

$$\int_0^b\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{p\pi y}{b}\,dy=\int_0^b xy(b-y)(a-x)\sin\frac{p\pi y}{b}\,dy$$

Bring the constants in front of the integrals.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\int_0^b\sin\frac{m\pi y}{b}\sin\frac{p\pi y}{b}\,dy=x(a-x)\int_0^b y(b-y)\sin\frac{p\pi y}{b}\,dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $m\neq p$. As a result, every term in the outer infinite series vanishes except for the $m=p$ one.

$$\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\int_0^b\sin^2\frac{m\pi y}{b}\,dy=x(a-x)\int_0^b y(b-y)\sin\frac{m\pi y}{b}\,dy$$

Evaluate the integrals.

$$\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\left(\frac{b}{2}\right)=x(a-x)\left[-\frac{2b^3[-1+(-1)^m]}{m^3\pi^3}\right]$$

Multiply both sides by $2/b$.

$$\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}=x(a-x)\left[-\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\right]$$

Multiply both sides by $\sin(q\pi x/a)$, where q is an integer,

$$\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{q\pi x}{a} = x(a-x)\left[-\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\right]\sin\frac{q\pi x}{a}$$

and then integrate both sides with respect to x from 0 to a.

$$\int_0^a \sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{q\pi x}{a}\,dx = \int_0^a x(a-x)\left[-\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\right]\sin\frac{q\pi x}{a}\,dx$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty} A_{mn} \int_0^a \sin\frac{n\pi x}{a} \sin\frac{q\pi x}{a}\,dx = -\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\int_0^a x(a-x)\sin\frac{q\pi x}{a}\,dx$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq q$. As a result, every term in the infinite series vanishes except for the $n = q$ one.

$$A_{mn} \int_0^a \sin^2\frac{n\pi x}{a}\,dx = -\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx$$

Evaluate the integrals.

$$A_{mn}\left(\frac{a}{2}\right) = -\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\left[-\frac{2a^3[-1+(-1)^n]}{n^3\pi^3}\right]$$

Multiply both sides by $2/a$ to solve for A_{mn}.

$$\begin{aligned} A_{mn} &= -\frac{4b^2[-1+(-1)^m]}{m^3\pi^3}\left[-\frac{4a^2[-1+(-1)^n]}{n^3\pi^3}\right] \\ &= \frac{16a^2b^2}{m^3n^3\pi^6}[-1+(-1)^m][-1+(-1)^n] \end{aligned}$$

Consequently, the general solution becomes

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{16a^2b^2}{m^3n^3\pi^6}[-1+(-1)^m][-1+(-1)^n]\cos\left(c\pi\sqrt{\frac{m^2}{b^2}+\frac{n^2}{a^2}}t\right)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}.$$

Notice that the summand vanishes if m or n are even. This answer can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $m = 2k-1$ and $n = 2l-1$ in the double series.

$$\begin{aligned} u(x,y,t) = \sum_{2k-1=1}^{\infty}\sum_{2l-1=1}^{\infty} &\frac{16a^2b^2}{(2k-1)^3(2l-1)^3\pi^6}(-2)(-2) \\ &\times \cos\left[c\pi\sqrt{\frac{(2k-1)^2}{b^2}+\frac{(2l-1)^2}{a^2}}t\right]\sin\frac{(2l-1)\pi x}{a}\sin\frac{(2k-1)\pi y}{b} \end{aligned}$$

Therefore,

$$u(x,y,t) = \frac{64a^2b^2}{\pi^6}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(2k-1)^3(2l-1)^3}\cos\left[c\pi\sqrt{\frac{(2k-1)^2}{b^2}+\frac{(2l-1)^2}{a^2}}t\right]\sin\frac{(2l-1)\pi x}{a}\sin\frac{(2k-1)\pi y}{b}.$$