Exercise 1

Solve the wave equation in the square $S = \{0 < x < \pi,\ 0 < y < \pi\}$, with homogeneous Neumann conditions on the boundary, and the initial conditions $u(x, y, 0) \equiv 0$, $u_t(x, y, 0) = \sin^2 x$.

Solution

The initial boundary value problem to solve is

$$\begin{gathered} u_{tt} = c^2\nabla^2 u, \quad 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\ u_x(0, y, t) = 0 \qquad u_y(x, 0, t) = 0 \\ u_x(\pi, y, t) = 0 \qquad u_y(x, \pi, t) = 0 \\ u(x, y, 0) = 0 \qquad u_t(x, y, 0) = \sin^2 x. \end{gathered}$$

Because the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve for u. Assume a product solution of the form $u(x, y, t) = X(x)Y(y)T(t)$ and substitute it into the PDE

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad \rightarrow \quad XYT'' = c^2(X''YT + XY''T)$$

and the boundary conditions.

$$\begin{aligned} u_x(0, y, t) = 0 \quad &\rightarrow \quad X'(0)Y(y)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ u_x(\pi, y, t) = 0 \quad &\rightarrow \quad X'(\pi)Y(y)T(t) = 0 \quad &\rightarrow \quad X'(\pi) = 0 \\ u_y(x, 0, t) = 0 \quad &\rightarrow \quad X(x)Y'(0)T(t) = 0 \quad &\rightarrow \quad Y'(0) = 0 \\ u_y(x, \pi, t) = 0 \quad &\rightarrow \quad X(x)Y'(\pi)T(t) = 0 \quad &\rightarrow \quad Y'(\pi) = 0 \end{aligned}$$

Separate variables in the PDE: divide both sides by c^2XYT so that the left side is a function of t and the right side is a function of x and y.

$$\underbrace{\frac{T''}{c^2T}}_{\text{function of } t} = \underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{function of } x \text{ and } y}$$

The only way a function of t can be equal to a function of x and y is if both are equal to a constant λ.

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Subtract both sides of the second equation by Y''/Y.

$$\underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\lambda - \frac{Y''}{Y}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to another constant μ.

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu$$

As a result of applying the method of separation of variables, the PDE has been reduced to a system of three ODEs—one in x, one in y, and one in t.

$$\left.\begin{aligned} \frac{T''}{c^2T} &= \lambda \\ \frac{X''}{X} &= \mu \\ \lambda - \frac{Y''}{Y} &= \mu \end{aligned}\right\}$$

Start by solving the ODE for X.

$$X'' = \mu X$$

Suppose first that μ is positive: $\mu = \alpha^2$.

$$X'' = \alpha^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Apply the boundary conditions for X to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= C_2\alpha = 0 \\ X'(\pi) &= \alpha(C_1 \sinh \alpha\pi + C_2 \cosh \alpha\pi) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second equation simplifies to $C_1\alpha \sinh \alpha\pi = 0$. No nonzero value of α satisfies this equation, so $C_1 = 0$. The trivial solution $X(x) = 0$ is obtained, which means there are no positive eigenvalues. Suppose now that μ is zero: $\mu = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_3$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(\pi) &= C_3 = 0 \end{aligned}$$

So then

$$X' = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Since $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With μ set to zero, the ODE for Y becomes

$$Y'' = \lambda Y.$$

There are (to be shown later) negative eigenvalues $\lambda = -n^2$ with $Y_n(y) = \cos ny$ for $n = 1, 2, \ldots$ and a zero eigenvalue $\lambda = 0$ with $Y_0(y) = 1$. Now solve the ODE for T with these values for λ. Beginning with $\lambda = 0$, the ODE for T becomes

$$T'' = 0 \quad \rightarrow \quad T(t) = C_7 t + C_8.$$

With $\lambda = -n^2$, the ODE for T becomes

$$T'' = -c^2 n^2 T \quad \rightarrow \quad T(t) = C_9 \cos cnt + C_{10} \sin cnt.$$

Suppose now that μ is negative: $\mu = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_{11} \cos \beta x + C_{12} \sin \beta x$$

Apply the boundary conditions for X to determine C_{11} and C_{12}.

$$\begin{aligned} X'(0) &= C_{12}\beta = 0 \\ X'(\pi) &= \beta(-C_{11} \sin \beta\pi + C_{12} \cos \beta\pi) = 0 \end{aligned}$$

The first equation implies that $C_{12} = 0$, so the second equation reduces to $-C_{11}\beta \sin \beta\pi = 0$. To avoid getting the trivial solution, we insist that $C_{11} \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta\pi &= 0 \\ \sin \beta\pi &= 0 \\ \beta\pi &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= n. \end{aligned}$$

The negative eigenvalues are $\mu = -n^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_{11} \cos \beta x + C_{12} \sin \beta x \\ &= C_{11} \cos \beta x \quad \rightarrow \quad X_n(x) = \cos nx. \end{aligned}$$

n is only a positive integer because negative values lead to redundant values of μ. With $\mu = -n^2$, the ODE for Y becomes

$$\begin{gathered} \lambda - \frac{Y''}{Y} = -n^2 \\ Y'' = -(-n^2 - \lambda)Y. \end{gathered}$$

Suppose first that λ is positive: $\lambda = \xi^2$. The ODE for Y becomes

$$Y'' = (n^2 + \xi^2)Y,$$

which has a solution in terms of hyperbolic sine and hyperbolic cosine. Like before, this leads to the trivial solution, so λ is not positive. Suppose secondly that λ is zero: $\lambda = 0$.

$$Y'' = n^2 Y$$

The trivial solution is obtained once again because Y is in terms of hyperbolic functions, so zero is not an eigenvalue for λ. Suppose thirdly that λ is negative and, in particular, equal to $-n^2$: $\lambda = -n^2$. The ODE for Y becomes

$$Y'' = 0 \quad \rightarrow \quad Y(y) = C_{13}.$$

With $\lambda = -n^2$, the ODE for T becomes

$$T'' = -c^2 n^2 T \quad \rightarrow \quad T(t) = C_{14} \cos cnt + C_{15} \sin cnt.$$

Suppose fourthly that λ is negative and not equal to $-n^2$: $\lambda = -\zeta^2 \neq -n^2$. The ODE for Y becomes

$$Y'' = -(\zeta^2 - n^2)Y.$$

The general solution is in terms of sine and cosine.

$$Y(y) = C_{16} \cos \sqrt{\zeta^2 - n^2}y + C_{17} \sin \sqrt{\zeta^2 - n^2}y$$

Apply the boundary conditions to determine C_{16} and C_{17}.

$$\begin{aligned} Y'(0) &= \sqrt{\zeta^2 - n^2} C_{17} = 0 \\ Y'(\pi) &= \sqrt{\zeta^2 - n^2}(-C_{16} \sin \sqrt{\zeta^2 - n^2}\pi + C_{17} \cos \sqrt{\zeta^2 - n^2}\pi) = 0 \end{aligned}$$

The first equation implies that $C_{17} = 0$, so the second equation reduces to $\sqrt{\zeta^2 - n^2}(-C_{16} \sin \sqrt{\zeta^2 - n^2}\pi) = 0$. To avoid getting the trivial solution, we insist that $C_{16} \neq 0$. Then

$$\begin{aligned} -\sqrt{\zeta^2 - n^2} \sin \sqrt{\zeta^2 - n^2}\pi &= 0 \\ \sin \sqrt{\zeta^2 - n^2}\pi &= 0 \\ \sqrt{\zeta^2 - n^2}\pi &= m\pi, \quad m = 1, 2, \ldots \\ \sqrt{\zeta^2 - n^2} &= m \\ \zeta^2 - n^2 &= m^2 \\ \zeta_{mn} &= \sqrt{m^2 + n^2}. \end{aligned}$$

The negative eigenvalues are $\lambda = -(m^2 + n^2)$, and the eigenfunctions associated with them are

$$\begin{aligned} Y(y) &= C_{16} \cos \sqrt{\zeta^2 - n^2}y + C_{17} \sin \sqrt{\zeta^2 - n^2}y \\ &= C_{16} \cos \sqrt{\zeta^2 - n^2}y \quad \rightarrow \quad Y_m(y) = \cos my. \end{aligned}$$

With $\lambda = -(m^2 + n^2)$, solve the ODE for T.

$$T'' = -c^2(m^2 + n^2)T \quad \rightarrow \quad T(t) = C_{18} \cos c\sqrt{m^2 + n^2}t + C_{19} \sin c\sqrt{m^2 + n^2}t$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)T(t)$ over all the eigenvalues.

$$\begin{aligned} u(x, y, t) = 1 \cdot &\left[1 \cdot (A_0 + B_0 t) + \sum_{n=1}^{\infty} (A_n \cos cnt + B_n \sin cnt) \cos ny\right] \\ &+ \sum_{n=1}^{\infty} \cos nx \left[1 \cdot (D_n \cos cnt + E_n \sin cnt) + \sum_{m=1}^{\infty} (F_{mn} \cos c\sqrt{m^2 + n^2}t + G_{mn} \sin c\sqrt{m^2 + n^2}t) \cos my\right] \end{aligned}$$

Simplify the right side.

$$\begin{aligned} u(x,y,t) = A_0 + B_0 t + \sum_{n=1}^{\infty}(A_n \cos cnt + B_n \sin cnt)\cos ny \\ + \sum_{n=1}^{\infty}(D_n \cos cnt + E_n \sin cnt)\cos nx \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(F_{mn}\cos c\sqrt{m^2+n^2}t + G_{mn}\sin c\sqrt{m^2+n^2}t)\cos my \cos nx \end{aligned}$$

The final task is to use the initial conditions, $u(x,y,0) = 0$ and $u_t(x,y,0) = \sin^2 x$, to determine the coefficients. Take a derivative of u with respect to t.

$$\begin{aligned} u_t(x,y,t) = B_0 + \sum_{n=1}^{\infty} cn(-A_n \sin cnt + B_n \cos cnt)\cos ny \\ + \sum_{n=1}^{\infty} cn(-D_n \sin cnt + E_n \cos cnt)\cos nx \\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{m^2+n^2}(-F_{mn}\sin c\sqrt{m^2+n^2}t + G_{mn}\cos c\sqrt{m^2+n^2}t)\cos my \cos nx \end{aligned}$$

Apply the initial conditions and use the fact that $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

$$\begin{aligned} u(x,y,0) &= A_0 + \sum_{n=1}^{\infty} A_n \cos ny + \sum_{n=1}^{\infty} D_n \cos nx + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\cos my \cos nx = 0 \\ u_t(x,y,0) &= B_0 + \sum_{n=1}^{\infty} cnB_n \cos ny + \sum_{n=1}^{\infty} cnE_n \cos nx + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{m^2+n^2}G_{mn}\cos my \cos nx = \frac{1}{2} - \frac{1}{2}\cos 2x \end{aligned}$$

By inspection we see that

$$\begin{aligned} A_0 &= 0 \\ A_n &= 0 \\ D_n &= 0 \\ F_{mn} &= 0 \\ B_0 &= \frac{1}{2} \\ B_n &= 0 \\ cnE_n &= \begin{cases} 0 & \text{if } n \neq 2 \\ -\frac{1}{2} & \text{if } n = 2 \end{cases} \quad \rightarrow \quad E_n = \begin{cases} 0 & \text{if } n \neq 2 \\ -\frac{1}{4c} & \text{if } n = 2 \end{cases} \\ G_{mn} &= 0. \end{aligned}$$

Therefore,

$$u(x,y,t) = \frac{t}{2} - \frac{1}{4c}\sin 2ct \cos 2x.$$

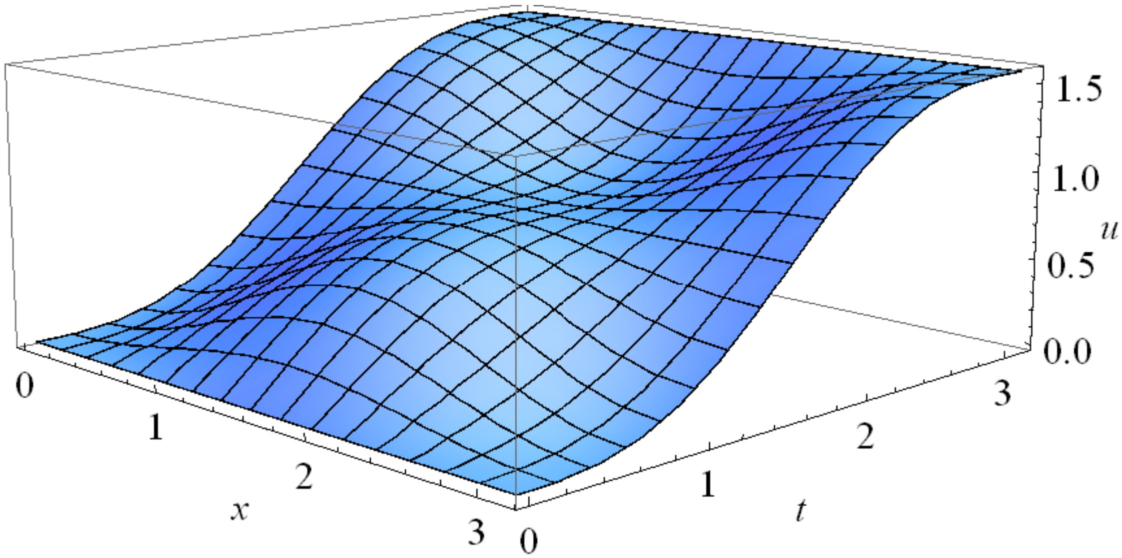

Figure 1: This figure illustrates the two-dimensional solution surface $u(x, y, t)$ living in three-dimensional space (x, t, u) for $c = 1$.