Exercise 2

Solve the wave equation in the rectangle $R = \{0 < x < a, \ 0 < y < b\}$, with homogeneous Dirichlet conditions on the boundary, and the initial conditions $u(x, y, 0) = xy(b - y)(a - x)$, $u_t(x, y, 0) \equiv 0$.

Solution

The initial boundary value problem to solve is

$$u_{tt} = c^2 \nabla^2 u, \quad 0 < x < a, \ 0 < y < b, \ t > 0$$

$$u(0, y, t) = 0 \quad u(x, 0, t) = 0$$

$$u(a, y, t) = 0 \quad u(x, b, t) = 0$$

$$u(x, y, 0) = xy(b - y)(a - x)$$

$$u_t(x, y, 0) = 0.$$

Because the PDE and its associated boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve for $u$. Assume a product solution of the form $u(x, y, t) = X(x)Y(y)T(t)$ and substitute it into the PDE

$$u_{tt} = c^2(u_{xx} + u_{yy}) \quad \rightarrow \quad XYT'' = c^2(X''Y + XY'')$$

and the boundary conditions.

$$u(0, y, t) = 0 \quad \rightarrow \quad X(0)Y(y)T(t) = 0 \quad \rightarrow \quad X(0) = 0$$

$$u(a, y, t) = 0 \quad \rightarrow \quad X(a)Y(y)T(t) = 0 \quad \rightarrow \quad X(a) = 0$$

$$u(x, 0, t) = 0 \quad \rightarrow \quad X(x)Y(0)T(t) = 0 \quad \rightarrow \quad Y(0) = 0$$

$$u(x, b, t) = 0 \quad \rightarrow \quad X(x)Y(b)T(t) = 0 \quad \rightarrow \quad Y(b) = 0$$

Separate variables in the PDE: divide both sides by $c^2XYT$ so that the left side is a function of $t$ and the right side is a function of $x$ and $y$.

$$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

The only way a function of $t$ can be equal to a function of $x$ and $y$ is if both are equal to a constant $\lambda$.

$$\frac{T''}{c^2 T} = \frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

Subtract both sides of the second equation by $Y''/Y$.

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

The only way a function of $x$ can be equal to a function of $y$ is if both are equal to another constant $\mu$.

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu$$

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As a result of applying the method of separation of variables, the PDE has been reduced to a system of three ODEs—one in \( x \), one in \( y \), and one in \( t \).

\[
\begin{align*}
\frac{T''}{c^2 T} &= \lambda \\
\frac{X''}{X} &= \mu \\
\lambda - \frac{Y''}{Y} &= \mu
\end{align*}
\]

Start by solving the ODE for \( X \).

\[ X'' = \mu X \]

Suppose first that \( \mu \) is positive: \( \mu = \alpha^2 \).

\[ X'' = \alpha^2 X \]

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

\[ X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x \]

Apply the boundary conditions for \( X \) to determine \( C_1 \) and \( C_2 \).

\[
\begin{align*}
X(0) &= C_1 = 0 \\
X(a) &= C_1 \cosh \alpha a + C_2 \sinh \alpha a = 0
\end{align*}
\]

The second equation reduces to \( C_2 \sinh \alpha a = 0 \). No nonzero value of \( \alpha \) satisfies this equation, so \( C_2 = 0 \). The trivial solution \( X(x) = 0 \) is obtained, which means there are no positive eigenvalues.

Suppose now that \( \mu \) is zero: \( \mu = 0 \). The ODE for \( X \) becomes

\[ X'' = 0 \]

The general solution is obtained by integrating both sides with respect to \( x \) twice.

\[ X(x) = C_3 x + C_4 \]

Apply the boundary conditions to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
X(0) &= C_4 = 0 \\
X(a) &= C_3 a + C_4 = 0
\end{align*}
\]

The second equation reduces to \( C_3 a = 0 \), so \( C_3 = 0 \). The trivial solution \( X(x) = 0 \) is obtained, which means zero is not an eigenvalue. Suppose now that \( \mu \) is negative: \( \mu = -\beta^2 \). The ODE for \( X \) becomes

\[ X'' = -\beta^2 X \]

The general solution is written in terms of sine and cosine.

\[ X(x) = C_5 \cos \beta x + C_6 \sin \beta x \]

Apply the boundary conditions to determine \( C_5 \) and \( C_6 \).

\[
\begin{align*}
X(0) &= C_5 = 0 \\
X(a) &= C_5 \cos \beta a + C_6 \sin \beta a = 0
\end{align*}
\]
The second equation reduces to $C_6 \sin \beta a = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Then

$$\sin \beta a = 0$$
$$\beta a = n\pi, \quad n = 1, 2, \ldots$$
$$\beta_n = \frac{n\pi}{a}.$$ 

There are negative eigenvalues $\mu = -\frac{n^2 \pi^2}{a^2}$, and the eigenfunctions associated with them are

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$
$$= C_6 \sin \beta x \quad \rightarrow \quad X_n(x) = \sin \frac{n\pi x}{a}.$$ 

With $\mu = -\frac{n^2 \pi^2}{a^2}$, the ODE for $Y$ becomes

$$\lambda - \frac{Y''}{Y} = -\frac{n^2 \pi^2}{a^2}$$
$$Y'' = -\left(-\lambda - \frac{n^2 \pi^2}{a^2}\right)Y.$$ 

Suppose first that $\lambda$ is positive: $\lambda = \xi^2$.

$$Y'' = \left(\xi^2 + \frac{n^2 \pi^2}{a^2}\right)Y.$$ 

The general solution is written in terms of hyperbolic sine and cosine.

$$Y(y) = C_7 \cosh \sqrt{\xi^2 + \frac{n^2 \pi^2}{a^2}} y + C_8 \sinh \sqrt{\xi^2 + \frac{n^2 \pi^2}{a^2}} y$$ 

Apply the boundary conditions to determine $C_7$ and $C_8$.

$$Y(0) = C_7 = 0$$
$$Y(b) = C_7 \cosh \sqrt{\xi^2 + \frac{n^2 \pi^2}{a^2}} b + C_8 \sinh \sqrt{\xi^2 + \frac{n^2 \pi^2}{a^2}} b = 0$$ 

The second equation reduces to $C_8 \sinh \sqrt{\xi^2 + \frac{n^2 \pi^2}{a^2}} b = 0$. No nonzero value of $\xi$ satisfies this equation, so $C_8$ must be zero. The trivial solution $Y(y) = 0$ is obtained, so there are no positive eigenvalues for $\lambda$. Suppose secondly that $\lambda$ is zero: $\lambda = 0$.

$$Y'' = \frac{n^2 \pi^2}{a^2}Y$$ 

Like before, the general solution is written in terms of hyperbolic sine and hyperbolic cosine, which leads to the trivial solution. Zero is not an eigenvalue. Suppose thirdly that $\lambda$ is negative: $\lambda = -\xi^2$. The ODE for $Y$ becomes

$$Y'' = -\left(\xi^2 - \frac{n^2 \pi^2}{a^2}\right)Y.$$ 

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The general solution is written in terms of sine and cosine.

\[ Y(y) = C_9 \cos \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} y + C_{10} \sin \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} y \]

Apply the boundary conditions to determine \( C_9 \) and \( C_{10} \).

\( Y(0) = C_9 = 0 \)
\( Y(b) = C_9 \cos \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} b + C_{10} \sin \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} b = 0 \)

The second equation reduces to \( C_{10} \sin \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} b = 0 \). To avoid getting the trivial solution, we insist that \( C_{10} \neq 0 \). Then

\[ \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} b = m\pi, \quad m = 1, 2, \ldots \]

There are negative eigenvalues \( \lambda = -\pi^2 \left( \frac{m^2}{b^2} + \frac{n^2}{a^2} \right) \), and the eigenfunctions associated with them are

\[ Y(y) = C_9 \cos \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} y + C_{10} \sin \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} y = C_{10} \sin \sqrt{\zeta^2 - \frac{n^2 \pi^2}{a^2}} y \rightarrow Y_m(y) = \sin \frac{m\pi y}{b}. \]

With \( \lambda = -\pi^2 \left( \frac{m^2}{b^2} + \frac{n^2}{a^2} \right) \), the ODE for \( T \) becomes

\[ T'' = -c^2 \pi^2 \left( \frac{m^2}{b^2} + \frac{n^2}{a^2} \right) T. \]

The general solution is written in terms of sine and cosine.

\[ T(t) = C_{11} \cos c \pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} t + C_{12} \sin c \pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} t \]

According to the principle of superposition, the general solution to the PDE for \( u \) is a linear combination of \( X(x)Y(y)T(t) \) over all the eigenvalues.

\[ u(x, y, t) = \sum_{n=1}^{\infty} \frac{n\pi x}{a} \left[ \sum_{m=1}^{\infty} \left( A_{mn} \cos c \pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} t + B_{mn} \sin c \pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} t \right) \sin \frac{m\pi y}{b} \right] \]
Simplify the right side.

\[ u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (A_{mn} \cos \frac{c\pi}{b} \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2} t} + B_{mn} \sin \frac{c\pi}{b} \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2} t}) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \]

The final task is to use the initial conditions, \( u(x, y, 0) = xy(b - y)(a - x) \) and \( u_t(x, y, 0) = 0 \), to determine the coefficients. Take a derivative of \( u \) with respect to \( t \).

\[ u_t(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c\pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} \left( -A_{mn} \sin \frac{c\pi}{b} \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2} t} + B_{mn} \cos \frac{c\pi}{b} \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2} t} \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \]

Apply the initial conditions.

\[ u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = xy(b - y)(a - x) \]

\[ u_t(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c\pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}} B_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = 0 \]

By inspection we see that \( B_{mn} = 0 \). To find \( A_{mn} \), multiply both sides of the first equation by \( \sin(p\pi y/b) \), where \( p \) is an integer,

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi y}{b} = xy(b - y)(a - x) \sin \frac{p\pi y}{b} \]

and then integrate both sides with respect to \( y \) from 0 to \( b \).

\[ \int_0^b \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{p\pi y}{b} \, dy = \int_0^b xy(b - y)(a - x) \sin \frac{p\pi y}{b} \, dy \]

Bring the constants in front of the integrals.

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \int_0^b \sin \frac{m\pi y}{b} \sin \frac{p\pi y}{b} \, dy = x(a - x) \int_0^b y(b - y) \sin \frac{p\pi y}{b} \, dy \]

Because the sine functions are orthogonal, the integral on the left is zero if \( m \neq p \). As a result, every term in the outer infinite series vanishes except for the \( m = p \) one.

\[ \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \int_0^b \sin^2 \frac{m\pi y}{b} \, dy = x(a - x) \int_0^b y(b - y) \sin \frac{m\pi y}{b} \, dy \]

Evaluate the integrals.

\[ \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \left( \frac{b}{2} \right) = x(a - x) \left[ -\frac{2b^3(-1 + (-1)^m)}{m^3\pi^3} \right] \]

Multiply both sides by \( 2/b \).

\[ \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} = x(a - x) \left[ -\frac{4b^2(-1 + (-1)^m)}{m^3\pi^3} \right] \]

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Multiply both sides by $\sin(q\pi x/a)$, where $q$ is an integer,

$$
\sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{q\pi x}{a} = x(a - x) \left[ -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \right] \sin \frac{q\pi x}{a}
$$

and then integrate both sides with respect to $x$ from 0 to $a$.

$$
\int_0^a \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{q\pi x}{a} \, dx = \int_0^a x(a - x) \left[ -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \right] \sin \frac{q\pi x}{a} \, dx
$$

Bring the constants in front of the integrals.

$$
\sum_{n=1}^{\infty} A_{mn} \int_0^a \sin \frac{n\pi x}{a} \sin \frac{q\pi x}{a} \, dx = -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \int_0^a x(a - x) \sin \frac{q\pi x}{a} \, dx
$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq q$. As a result, every term in the infinite series vanishes except for the $n = q$ one.

$$
A_{mn} \int_0^a \sin^2 \frac{n\pi x}{a} \, dx = -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \int_0^a x(a - x) \sin \frac{q\pi x}{a} \, dx
$$

Evaluate the integrals.

$$
A_{mn} \left( \frac{a}{2} \right) = -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \left[ -\frac{2a^3[-1 + (-1)^n]}{n^3\pi^3} \right]
$$

Multiply both sides by $2/a$ to solve for $A_{mn}$.

$$
A_{mn} = -\frac{4b^2[-1 + (-1)^m]}{m^3\pi^3} \left[ -\frac{4a^2[-1 + (-1)^n]}{n^3\pi^3} \right] = \frac{16a^2b^2}{m^3n^3\pi^6}[-1 + (-1)^m][-1 + (-1)^n]
$$

Consequently, the general solution becomes

$$
u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{16a^2b^2}{m^3n^3\pi^6}[-1 + (-1)^m][-1 + (-1)^n] \cos \left( c\pi \sqrt{\frac{m^2}{b^2} + \frac{n^2}{a^2}t} \right) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}.
$$

Notice that the summand vanishes if $m$ or $n$ are even. This answer can thus be simplified (that is, made to converge faster) by summing over the odd integers only. Let $m = 2k - 1$ and $n = 2l - 1$ in the double series.

$$
u(x, y, t) = \sum_{2k-1=1}^{\infty} \sum_{2l-1=1}^{\infty} \frac{16a^2b^2}{(2k - 1)^3(2l - 1)^3\pi^6}(-2)(-2)
$$

$$
\times \cos \left[ c\pi \sqrt{\frac{(2k - 1)^2}{b^2} + \frac{(2l - 1)^2}{a^2}t} \right] \sin \frac{(2l - 1)\pi x}{a} \sin \frac{(2k - 1)\pi y}{b}
$$

Therefore,

$$
u(x, y, t) = \frac{64a^2b^2}{\pi^6} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{(2k - 1)^3(2l - 1)^3} \cos \left[ c\pi \sqrt{\frac{(2k - 1)^2}{b^2} + \frac{(2l - 1)^2}{a^2}t} \right] \sin \frac{(2l - 1)\pi x}{a} \sin \frac{(2k - 1)\pi y}{b}.
$$

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