Exercise 5

Find the dimension of each of the following vector spaces.

(a) The space of all the solutions of $u'' + x^2 u = 0$.

(b) The eigenspace with eigenvalue $(2\pi/l)^2$ of the operator $-d^2/dt^2$ on the interval $(-l, l)$ with the periodic boundary conditions.

(c) The space of harmonic functions in the unit disk with the homogeneous Neumann BCs.

(d) The eigenspace with eigenvalue $\lambda = 25\pi^2$ of $-\Delta$ in the unit square $(0,1)^2$ with the homogeneous Neumann BCs on all four sides.

(e) The space of all the solutions of $u_{tt} = c^2 u_{xx}$ in $-\infty < x < \infty$, $-\infty < t < \infty$.

Solution

Part (a)

Because the ODE is linear and its order is 2, it has two linearly independent solutions. The dimension of the solution space is 2.

Part (b)

The eigenvalue problem here is

$$\begin{aligned} -\frac{d^2u}{dt^2} &= \lambda u, \quad -l < t < l \\ u(-l) &= u(l) \\ u'(-l) &= u'(l). \end{aligned}$$

The eigenfunction associated with the eigenvalue $\lambda = 4\pi^2/l^2$ is

$$u(t) = C_1 \cos\frac{2\pi t}{l} + C_2 \sin\frac{2\pi t}{l},$$

where C_1 and C_2 remain arbitrary. Because there are two linearly independent solutions, $\cos\frac{2\pi t}{l}$ and $\sin\frac{2\pi t}{l}$, the dimension of the solution space is 2.

Part (c)

Here we're interested in the solution space to

$$\begin{aligned} &\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \le 1,\ 0 \le \theta \le 2\pi \\ &u_r(1,\theta) = 0. \end{aligned}$$

The general solution to the Laplace equation on this domain is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

The boundary condition will now be applied to determine the coefficients. Take a derivative of u with respect to r.

$$u_r(r,\theta) = \sum_{n=1}^{\infty} nr^{n-1}(A_n \cos n\theta + B_n \sin n\theta)$$

Then

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta) = 0.$$

By inspection we see that $A_n = 0$ and $B_n = 0$. The general solution reduces to

$$u(r,\theta) = A_0,$$

where A_0 remains arbitrary. Since there's one arbitrary constant, the dimension of the solution space is 1.

Part (d)

Here we wish to find the eigenfunction that satisfies

$$\begin{gathered} -\Delta u = 25\pi^2 u, \quad 0 < x < 1, \ 0 < y < 1 \\ u_x(0,y) = 0 \qquad u_y(x,0) = 0 \\ u_x(1,y) = 0 \qquad u_y(x,1) = 0. \end{gathered}$$

Note that this PDE is known as the Helmholtz equation. Because it and the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x,y) = X(x)Y(y)$ and substitute it into the PDE

$$-(u_{xx} + u_{yy}) = 25\pi^2 u \quad \to \quad -(X''Y + XY'') = 25\pi^2 XY$$

and the boundary conditions.

$$\begin{aligned} u_x(0,y) &= 0 & &\to & X'(0)Y(y) &= 0 & &\to & X'(0) &= 0 \\ u_x(1,y) &= 0 & &\to & X'(1)Y(y) &= 0 & &\to & X'(1) &= 0 \\ u_y(x,0) &= 0 & &\to & X(x)Y'(0) &= 0 & &\to & Y'(0) &= 0 \\ u_y(x,1) &= 0 & &\to & X(x)Y'(1) &= 0 & &\to & Y'(1) &= 0 \end{aligned}$$

Divide both sides by $-XY$ in the PDE in order to separate variables.

$$\frac{X''}{X} + \frac{Y''}{Y} = -25\pi^2$$

Bring Y''/Y to the right side.

$$\underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{-25\pi^2 - \frac{Y''}{Y}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant μ.

$$\frac{X''}{X} = -25\pi^2 - \frac{Y''}{Y} = \mu$$

As a result of applying the method of separation of variables, the PDE has been reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{X''}{X} &= \mu \\ -25\pi^2 - \frac{Y''}{Y} &= \mu \end{aligned}\right\}$$

Start by solving the ODE for X.

$$X'' = \mu X$$

Suppose first that μ is positive: $\mu = \alpha^2$.

$$X'' = \alpha^2 X$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative of it.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(1) &= \alpha(C_1 \sinh \alpha + C_2 \cosh \alpha) = 0 \end{aligned}$$

The first equation implies that $C_2 = 0$, so the second one reduces to $C_1 \alpha \sinh \alpha = 0$. No nonzero value of α satisfies it, so C_1 must be zero. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that μ is zero: $\mu = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_3$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(1) &= C_3 = 0 \end{aligned}$$

Integrate both sides of the previous equation with respect to x once more.

$$X(x) = C_4$$

Because $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With $\mu = 0$, solve the ODE for Y now.

$$-25\pi^2 - \frac{Y''}{Y} = 0$$

$$Y'' = -25\pi^2 Y$$

The general solution is in terms of sine and cosine.

$$Y(y) = C_5 \cos 5\pi y + C_6 \sin 5\pi y$$

Take a derivative of it.

$$Y'(y) = 5\pi(-C_5 \sin 5\pi y + C_6 \cos 5\pi y)$$

Apply the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} Y'(0) &= 5\pi(C_6) = 0 \\ Y'(1) &= 5\pi(-C_5 \sin 5\pi + C_6 \cos 5\pi) = 0 \end{aligned}$$

The first equation implies that $C_6 = 0$, so the second one reduces to $-5\pi C_5 \sin 5\pi = 0$, which is automatically satisfied.

$$Y(y) = C_5 \cos 5\pi y$$

Suppose thirdly that μ is negative: $\mu = -\beta^2$.

$$X'' = -\beta^2 X$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Take a derivative of it.

$$X'(x) = \beta(-C_7 \sin \beta x + C_8 \cos \beta x)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \beta(C_8) = 0 \\ X'(1) &= \beta(-C_7 \sin \beta + C_8 \cos \beta) = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $-C_7 \beta \sin \beta = 0$. To avoid the trivial solution, we insist that $C_7 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta &= 0 \\ \sin \beta &= 0 \\ \beta_n &= n\pi, \quad n = 1, 2, \ldots. \end{aligned}$$

There are negative eigenvalues $\mu = -n^2\pi^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_7 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos n\pi x. \end{aligned}$$

With $\mu = -n^2\pi^2$, solve the ODE for Y now.

$$-25\pi^2 - \frac{Y''}{Y} = -n^2\pi^2$$

$$Y'' = -(25 - n^2)\pi^2 Y$$

If $n = 5$, then

$$Y'' = 0 \quad \rightarrow \quad Y(y) = C_9.$$

If $0 < n < 5$, then

$$Y(y) = C_{10} \cos \sqrt{25 - n^2}\pi y + C_{11} \sin \sqrt{25 - n^2}\pi y.$$

($n > 5$ is irrelevant because it leads to hyperbolic functions, which lead to the trivial solution.) Take a derivative of it.

$$Y'(y) = \sqrt{25 - n^2}\pi(-C_{10} \sin \sqrt{25 - n^2}\pi y + C_{11} \cos \sqrt{25 - n^2}\pi y)$$

Apply the boundary conditions to determine C_{10} and C_{11}.

$$\begin{aligned} Y'(0) &= \sqrt{25 - n^2}\pi(C_{11}) = 0 \\ Y'(1) &= \sqrt{25 - n^2}\pi(-C_{10} \sin \sqrt{25 - n^2}\pi + C_{11} \cos \sqrt{25 - n^2}\pi) = 0 \end{aligned}$$

The first equation implies that $C_{11} = 0$, so the second one reduces to $-C_{10}\sqrt{25 - n^2}\pi \sin \sqrt{25 - n^2}\pi = 0$. Because $0 < n < 5$, this equation is only satisfied if $n = 3$ or $n = 4$.

$$\begin{aligned} n = 3: &\quad \cos 3\pi x \quad \text{and} \quad \cos 4\pi y \\ n = 4: &\quad \cos 4\pi x \quad \text{and} \quad \cos 3\pi y \end{aligned}$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$u(x, y) = 1 \cdot A \cos 5\pi y + \cos 5\pi x \cdot B(1) + \cos 4\pi x \cdot D \cos 3\pi y + \cos 3\pi x \cdot E \cos 4\pi y$$

Therefore,

$$u(x, y) = A \cos 5\pi y + B \cos 5\pi x + D \cos 4\pi x \cos 3\pi y + E \cos 3\pi x \cos 4\pi y.$$

Since there are four arbitrary constants, the dimension of the solution space is 4.

Part (e)

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \ -\infty < t < \infty$$

The general solution is

$$u(x, t) = F(x - ct) + G(x + ct),$$

where F and G are arbitrary functions. Since there are infinitely many linearly independent solutions to the wave equation, the dimension of the solution space is ∞.