Exercise 1

Verify each entry in the table of Fourier transforms. (Use (15) as needed.)

Solution

This is the table of Fourier transforms the exercise is referring to (on page 345).

		$f(x)$	$F(k)$
(5)	Delta function	$\delta(x)$	1
(6)	Square pulse	$H(a-\lvert x\rvert)$	$\frac{2}{k}\sin ak$
(7)	Exponential	$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}\quad (a>0)$
(8)	Heaviside function	$H(x)$	$\pi\delta(k)+\frac{1}{ik}$
(9)	Sign	$H(x)-H(-x)$	$\frac{2}{ik}$
(10)	Constant	1	$2\pi\delta(k)$
(11)	Gaussian	$e^{-x^2/2}$	$\sqrt{2\pi}e^{-k^2/2}$

We will verify these transforms using the definitions of the Fourier transform,

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx,$$

and inverse Fourier transform,

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk.$$

<u>Verification of Transform 5</u>

$$\begin{aligned}\mathcal{F}\{\delta(x)\} &= \int_{-\infty}^{\infty}\delta(x)e^{-ikx}\,dx \\ &= e^{-ik(0)} \\ &= 1\end{aligned}$$

This verifies the fifth transform.

<u>Verification of Transform 6</u>

$$\mathcal{F}\left\{H(a-|x|)\right\} = \int_{-\infty}^{\infty} H(a-|x|)\,e^{-ikx}\,dx$$

Using the definition of the Heaviside function $H(x)$, we can determine what effect $H(a-|x|)$ has on the integral.

$$H(x) = \begin{cases}0 & x<0\\ 1 & x>0\end{cases} \quad\to\quad H(a-|x|) = \begin{cases}0 & a-|x|<0\\ 1 & a-|x|>0\end{cases}$$

Consequently, the integrand is 0 when

$$\begin{aligned} a - |x| &< 0 \\ |x| &> a \\ x > a \quad \text{or} &\quad x < -a, \end{aligned}$$

and the integrand is e^{-ikx} when

$$\begin{aligned} a - |x| &> 0 \\ |x| &< a \\ -a < x &< a. \end{aligned}$$

So the integral simplifies to

$$\begin{aligned} \mathcal{F}\{H(a - |x|)\} &= \int_{-a}^{a} e^{-ikx}\, dx \\ &= \frac{1}{-ik} e^{-ikx}\bigg|_{-a}^{a} \\ &= \frac{1}{-ik}\left(e^{-ika} - e^{ika}\right) \\ &= \frac{2}{k}\frac{e^{ika} - e^{-ika}}{2i} \\ &= \frac{2}{k}\sin ak. \end{aligned}$$

This verifies the sixth transform.

<u>Verification of Transform 7</u>

$$\begin{aligned} \mathcal{F}\left\{e^{-a|x|}\right\} &= \int_{-\infty}^{\infty} e^{-a|x|}\, e^{-ikx}\, dx \\ &= \int_{-\infty}^{\infty} e^{-a|x|-ikx}\, dx \\ &= \int_{-\infty}^{0} e^{ax-ikx}\, dx + \int_{0}^{\infty} e^{-ax-ikx}\, dx \\ &= \int_{-\infty}^{0} e^{(a-ik)x}\, dx + \int_{0}^{\infty} e^{-(a+ik)x}\, dx \\ &= \frac{1}{a-ik} e^{(a-ik)x}\bigg|_{-\infty}^{0} + \frac{1}{-(a+ik)} e^{-(a+ik)x}\bigg|_{0}^{\infty} \\ &= \frac{1}{a-ik} + \frac{1}{(a+ik)} \\ &= \frac{2a}{a^2+k^2} \end{aligned}$$

This verifies the seventh transform.

Verification of Transform 8

Take the inverse Fourier transform of the entry on the right side and show that it yields the entry on the left side.

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\pi\delta(k)+\frac{1}{ik}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\pi\delta(k)+\frac{1}{ik}\right]e^{ikx}\,dk \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\delta(k)e^{ikx}\,dk+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{ik}\,dk \\
&= \frac{1}{2}e^{i(0)x}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos kx+i\sin kx}{ik}\,dk \\
&= \frac{1}{2}+\frac{1}{2\pi}\left(\frac{1}{i}\underbrace{\int_{-\infty}^{\infty}\frac{\cos kx}{k}\,dk}_{=\,0}+\int_{-\infty}^{\infty}\frac{\sin kx}{k}\,dk\right)
\end{aligned}$$

Because $(\cos kx)/k$ is an odd function of k, the integral of it from $-\infty$ to ∞ is equal to zero. The second integral is equal to $-\pi$ or π, depending on whether x is negative or positive, respectively.

$$\begin{aligned}
&= \frac{1}{2}+\frac{1}{2\pi}(\pi\operatorname{sgn}x) = \frac{1}{2}+\frac{1}{2}\operatorname{sgn}x = \begin{cases}0 & x<0\\ 1 & x>0\end{cases} \\
&= H(x)
\end{aligned}$$

This verifies the eighth transform.

Verification of Transform 9

Take the inverse Fourier transform of the entry on the right side and show that it yields the entry on the left side.

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\frac{2}{ik}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{ik}e^{ikx}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{ik}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos kx+i\sin kx}{ik}\,dk \\
&= \frac{1}{\pi}\left(\frac{1}{i}\underbrace{\int_{-\infty}^{\infty}\frac{\cos kx}{k}\,dk}_{=\,0}+\int_{-\infty}^{\infty}\frac{\sin kx}{k}\,dk\right)
\end{aligned}$$

Because $(\cos kx)/k$ is an odd function of k, the integral of it from $-\infty$ to ∞ is equal to zero. The second integral is equal to $-\pi$ or π, depending on whether x is negative or positive, respectively.

$$\begin{aligned}
&= \frac{1}{\pi}(\pi\operatorname{sgn}x) \\
&= \operatorname{sgn}x = \begin{cases}-1 & x<0\\ 1 & x>0\end{cases} \\
&= H(x)-H(-x)
\end{aligned}$$

This verifies the ninth transform.

Verification of Transform 10

Take the inverse Fourier transform of the entry on the right side and show that it yields the entry on the left side.

$$\begin{aligned}\mathcal{F}^{-1}\{2\pi\delta(k)\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(k)e^{ikx}\,dk \\ &= \int_{-\infty}^{\infty} \delta(k)e^{ikx}\,dk \\ &= e^{i(0)x} \\ &= 1\end{aligned}$$

This verifies the tenth transform.

Verification of Transform 11

$$\begin{aligned}\mathcal{F}\{e^{-x^2/2}\} &= \int_{-\infty}^{\infty} e^{-x^2/2}e^{-ikx}\,dx \\ &= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} - ikx\right)dx \\ &= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x^2 + 2ikx)\right]dx\end{aligned}$$

Complete the square in the exponent.

$$\begin{aligned}&= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x^2 + 2ikx - k^2) - \frac{k^2}{2}\right]dx \\ &= \exp\left(-\frac{k^2}{2}\right)\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x + ik)^2\right]dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned} y &= \frac{x+ik}{\sqrt{2}} &\quad\rightarrow\quad& y^2 = \frac{(x+ik)^2}{2} \\ dy &= \frac{dx}{\sqrt{2}} &\quad\rightarrow\quad& \sqrt{2}\,dy = dx\end{aligned}$$

As a result,

$$\begin{aligned}\mathcal{F}\{e^{-x^2/2}\} &= \exp\left(-\frac{k^2}{2}\right)\int_{-\infty}^{\infty} e^{-y^2}(\sqrt{2}\,dy) \\ &= \sqrt{2}\exp\left(-\frac{k^2}{2}\right)\int_{-\infty}^{\infty} e^{-y^2}\,dy \\ &= \sqrt{2\pi}\exp\left(-\frac{k^2}{2}\right)\end{aligned}$$

This verifies the eleventh transform.