

## Exercise 2

Verify each entry in the table of properties of Fourier transforms.

### Solution

This is the table of properties of Fourier transforms the exercise is referring to (on page 346).

	Function	Transform
(i)	$\frac{df}{dx}$	$ikF(k)$
(ii)	$xf(x)$	$i\frac{dF}{dk}$
(iii)	$f(x-a)$	$e^{-iak}F(k)$
(iv)	$e^{iax}f(x)$	$F(k-a)$
(v)	$af(x) + bg(x)$	$aF(k) + bG(k)$
(vi)	$f(ax)$	$\frac{1}{ a }F\left(\frac{k}{a}\right)$ ( $a \neq 0$ )

We will verify these transforms using the definitions of the Fourier transform,

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx} dx,$$

and inverse Fourier transform,

$$\mathcal{F}^{-1}\{F(k)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{ikx} dk.$$

### Verification of Property i

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{\infty} \frac{df}{dx} e^{-ikx} dx$$

Use integration by parts.

$$= f(x)e^{-ikx} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \frac{d}{dx}(e^{-ikx}) dx$$

$f(x)$  is assumed to tend to zero as  $x \rightarrow \pm\infty$ , so the first term is zero.

$$\begin{aligned} &= - \int_{-\infty}^{\infty} f(x)(-ik)e^{-ikx} dx \\ &= ik \int_{-\infty}^{\infty} f(x)e^{-ikx} dx \\ &= ikF(k) \end{aligned}$$

This verifies the first property.

Verification of Property ii

$$\begin{aligned}
\mathcal{F}\{xf(x)\} &= \int_{-\infty}^{\infty} xf(x)e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} xf(x) \frac{d}{dk} \left( \frac{e^{-ikx}}{-ix} \right) dx \\
&= \frac{1}{-i} \int_{-\infty}^{\infty} f(x) \frac{d}{dk} (e^{-ikx}) dx \\
&= i \frac{d}{dk} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\
&= i \frac{dF}{dk}
\end{aligned}$$

This verifies the second property.

Verification of Property iii

$$\mathcal{F}\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a)e^{-ikx} dx$$

Make the following substitution.

$$\begin{aligned}
y &= x - a \quad \rightarrow \quad y + a = x \\
dy &= dx
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathcal{F}\{f(x-a)\} &= \int_{-\infty}^{\infty} f(y)e^{-ik(y+a)} dx \\
&= \int_{-\infty}^{\infty} f(y)e^{-iky-ika} dx \\
&= e^{-ika} \int_{-\infty}^{\infty} f(y)e^{-iky} dx \\
&= e^{-iak} F(k).
\end{aligned}$$

This verifies the third property.

Verification of Property iv

$$\begin{aligned}
\mathcal{F}\{e^{iax}f(x)\} &= \int_{-\infty}^{\infty} e^{iax}f(x)e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-ikx+iax} dx = \int_{-\infty}^{\infty} f(x)e^{-i(k-a)x} dx \\
&= F(k-a)
\end{aligned}$$

This verifies the fourth property.

Verification of Property v

$$\begin{aligned}
\mathcal{F}\{af(x) + bg(x)\} &= \int_{-\infty}^{\infty} [af(x) + bg(x)]e^{-ikx} dx \\
&= \int_{-\infty}^{\infty} [af(x)e^{-ikx} + bg(x)e^{-ikx}] dx \\
&= \int_{-\infty}^{\infty} af(x)e^{-ikx} dx + \int_{-\infty}^{\infty} bg(x)e^{-ikx} dx \\
&= a \int_{-\infty}^{\infty} f(x)e^{-ikx} dx + b \int_{-\infty}^{\infty} g(x)e^{-ikx} dx \\
&= aF(k) + bG(k)
\end{aligned}$$

This verifies the fifth property.

Verification of Property vi

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{-ikx} dx$$

Make the following substitution.

$$\begin{aligned}
y = ax &\quad \rightarrow \quad \frac{y}{a} = x \\
dy = a dx &\quad \rightarrow \quad \frac{dy}{a} = dx
\end{aligned}$$

The sign of  $a$  does not influence the sign of the Fourier transform, so we use  $|a|$  in the formula for  $dx$ .

$$\begin{aligned}
\mathcal{F}\{f(ax)\} &= \int_{-\infty}^{\infty} f(y)e^{-ik\frac{y}{a}} \left(\frac{dy}{|a|}\right) \\
&= \frac{1}{|a|} \int_{-\infty}^{\infty} f(y)e^{-i\frac{k}{a}y} dy \\
&= \frac{1}{|a|} F\left(\frac{k}{a}\right)
\end{aligned}$$

This verifies the sixth property.