Exercise 3

Use the Fourier transform to find the bounded solution of the equation $-\Delta u + m^2 u = \delta(\mathbf{x})$ in free three-dimensional space with $m > 0$.

Solution

Δ is the Laplacian operator (∇^2), so the PDE to solve here is

$$-(u_{xx} + u_{yy} + u_{zz}) + m^2 u = \delta(x, y, z), \quad -\infty < x, y, z < \infty.$$

Since the PDE is linear and the x, y, and z variables go from $-\infty$ to ∞, the triple Fourier transform can be applied to solve it. Here we define the triple Fourier transform of a function $u(x, y, z)$ as

$$\mathcal{F}\{u(x, y, z)\} = U(k, l, M) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z) e^{-i(kx+ly+Mz)} \, dx \, dy \, dz.$$

As a result, the derivatives of u with respect to x and y and z transform as follows.

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} &= (ik)^n U(k, l, M) \\
\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} &= (il)^n U(k, l, M) \\
\mathcal{F}\left\{\frac{\partial^n u}{\partial z^n}\right\} &= (iM)^n U(k, l, M)
\end{aligned}$$

Take the triple Fourier transform of both sides of the PDE.

$$\mathcal{F}\{-(u_{xx} + u_{yy} + u_{zz}) + m^2 u\} = \mathcal{F}\{\delta(x, y, z)\}$$

Use the fact that the operator is linear on the left side and apply the definition on the right side.

$$-\mathcal{F}\{u_{xx}\} - \mathcal{F}\{u_{yy}\} - \mathcal{F}\{u_{zz}\} + m^2 \mathcal{F}\{u\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y, z) e^{-i(kx+ly+Mz)} \, dx \, dy \, dz$$

Transform the derivatives with the expressions above.

$$-(ik)^2 U(k, l, M) - (il)^2 U(k, l, M) - (iM)^2 U(k, l, M) + m^2 U(k, l, M) = e^{-i(k\cdot 0 + l\cdot 0 + M\cdot 0)}$$

Factor the left side.

$$(k^2 + l^2 + M^2 + m^2) U(k, l, M) = 1$$

Thus,

$$U(k, l, M) = \frac{1}{k^2 + l^2 + M^2 + m^2}.$$

Now that we have $U(k, l, M)$, we can change back to $u(x, y, z)$ by taking the inverse triple Fourier transform of it.

$$\begin{aligned}
u(x, y, z) &= \mathcal{F}^{-1}\{U(k, l, M)\} \\
&= \mathcal{F}^{-1}\left\{\frac{1}{k^2 + l^2 + M^2 + m^2}\right\} \\
&= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k^2 + l^2 + M^2 + m^2} e^{i(kx+ly+Mz)} \, dk \, dl \, dM
\end{aligned}$$

In order to solve this triple integral, we will switch to spherical coordinates with $\rho^2 = k^2 + l^2 + M^2$ and the polar axis pointing in the direction of $\mathbf{r} = \langle x, y, z \rangle$ so that $\boldsymbol{\rho} \cdot \mathbf{r} = kx + ly + Mz = \rho r \cos\phi$.

$$\begin{aligned} u(x,y,z) &= \frac{1}{(2\pi)^3} \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{1}{\rho^2 + m^2} e^{i\rho r\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\ &= \frac{1}{8\pi^3} \left(\int_0^{2\pi} d\theta \right) \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left(\int_0^\pi e^{i\rho r \cos\phi} \sin\phi \, d\phi \right) d\rho \\ &= \frac{1}{4\pi^2} \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left(\int_0^\pi e^{i\rho r\cos\phi} \sin\phi \, d\phi \right) d\rho \end{aligned}$$

Use a substitution to solve the integral in parentheses.

$$\begin{aligned} s &= i\rho r \cos\phi \\ ds &= i\rho r(-\sin\phi)\, d\phi \quad \rightarrow \quad -\frac{ds}{i\rho r} = \sin\phi \, d\phi \end{aligned}$$

Consequently,

$$\begin{aligned} u(x,y,z) &= \frac{1}{4\pi^2} \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left[\int_{i\rho r}^{-i\rho r} e^s \left(-\frac{ds}{i\rho r} \right) \right] d\rho \\ &= \frac{1}{4\pi^2 r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} \left(\int_{-i\rho r}^{i\rho r} \frac{e^s}{i} \right) d\rho \\ &= \frac{1}{4\pi^2 r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} \left(\frac{e^{i\rho r} - e^{-i\rho r}}{i} \right) d\rho \\ &= \frac{1}{4\pi^2 r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} (2\sin\rho r)\, d\rho. \end{aligned}$$

The integrand is even in ρ, so the factor of 2 can be used to extend the interval of integration to $(-\infty, \infty)$.

$$u(x,y,z) = \frac{1}{4\pi^2 r} \int_{-\infty}^\infty \frac{\rho}{\rho^2 + m^2} \sin\rho r \, d\rho$$

The theory of residues will be used to evaluate the final integral. Consider the function,

$$\begin{aligned} f(z) &= \frac{z}{z^2 + m^2} \\ &= \frac{z}{(z + im)(z - im)}, \end{aligned}$$

where z is a complex number. $f(z)e^{irz}$ is analytic everywhere in the complex plane except at the singularities, $z = -im$ and $z = im$. $z = im$ lies inside the semicircular region shown in Figure 1.

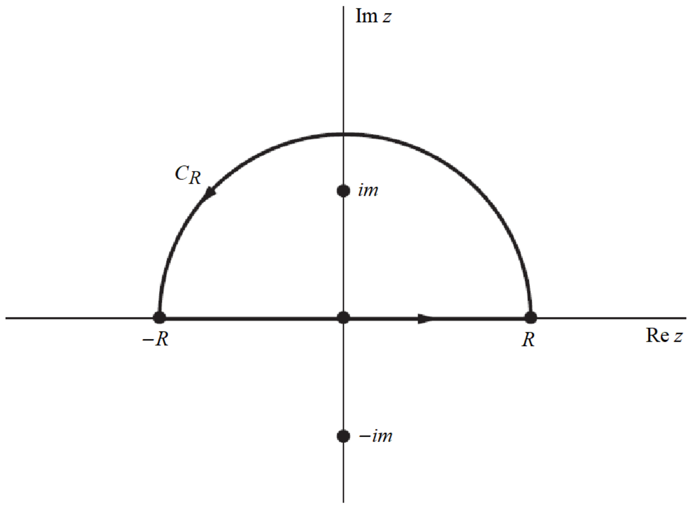

Figure 1: One of the singularities of $f(z)e^{irz}$, $z = im$, lies inside the semicircle, so the residue at this point will have to be evaluated.

Integrating $f(z)e^{irz}$ around the boundary C of the semicircle in Figure 1 gives us

$$\int_C f(z)e^{irz}\,dz = \int_{-R}^{R} f(z)e^{irz}\,dz + \int_{C_R} f(z)e^{irz}\,dz,$$

which is equal to $2\pi i$ times the residue at $z = im$ according to Cauchy's residue theorem.

$$\int_{-R}^{R} f(z)e^{irz}\,dz + \int_{C_R} f(z)e^{irz}\,dz = 2\pi i \operatorname*{Res}_{z=im} f(z)e^{irz}$$

Because $z = im$ is a simple pole, we have

$$\operatorname*{Res}_{z=im} f(z)e^{irz} = \phi(im), \quad \text{where} \quad \phi(z) = \frac{z}{(z+im)}e^{irz}.$$

As a result,

$$\int_{-R}^{R} f(z)e^{irz}\,dz + \int_{C_R} f(z)e^{irz}\,dz = 2\pi i \cdot \frac{im}{2im}e^{-mr} = i\pi e^{-mr}.$$

Now take the limit as $R \to \infty$.

$$\int_{-\infty}^{\infty} f(z)e^{irz}\,dz + \lim_{R\to\infty}\int_{C_R} f(z)e^{irz}\,dz = i\pi e^{-mr}$$

The limit of the second integral is equal to zero by Jordan's lemma because

$$\left|\int_{C_R} f(z)e^{irz}\,dz\right| \le \int_{C_R} |f(z)e^{irz}|\,dz \le \pi R \cdot \frac{R}{R^2+m^2} \quad \text{and} \quad \lim_{R\to\infty}\frac{R}{R^2+m^2} = 0.$$

What remains then is

$$\int_{-\infty}^{\infty} f(z)e^{irz}\,dz = i\pi e^{-mr}.$$

Use Euler's formula here.

$$\int_{-\infty}^{\infty} f(z)(\cos rz + i\sin rz)\,dz = i\pi e^{-mr}$$

Write the integral as the sum of a real part and an imaginary part.

$$\int_{-\infty}^{\infty} f(z)\cos rz\,dz + i\int_{-\infty}^{\infty} f(z)\sin rz\,dz = i\pi e^{-mr}$$

From this equation, by matching the real and imaginary parts on both sides, we deduce the following integral formulas.

$$\int_{-\infty}^{\infty} \frac{z}{z^2+m^2}\cos rz\,dz = 0$$
$$\int_{-\infty}^{\infty} \frac{z}{z^2+m^2}\sin rz\,dz = \pi e^{-mr}$$

Use the second formula in the equation for $u(x, y, z)$.

$$\begin{aligned} u(x,y,z) &= \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty} \frac{\rho}{\rho^2+m^2}\sin\rho r\,d\rho \\ &= \frac{1}{4\pi^2 r}\cdot \pi e^{-mr} \\ &= \frac{1}{4\pi r}e^{-mr} \end{aligned}$$

Therefore,

$$u(x,y,z) = \frac{e^{-m\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}.$$