Exercise 4

If $p(x)$ is a polynomial and $f(x)$ is any continuous function on the interval $[a,b]$, show that $g(x) = \int_a^b p(x-s)f(s)\,ds$ is also a polynomial.

Solution

Suppose $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ is a polynomial. Then it can be represented as a series.

$$p(x) = \sum_{n=0}^{N} c_n x^n$$

Our aim is to show that $g(x)$ can be represented similarly. Substitute the series for $p(x)$ into the formula for $g(x)$.

$$g(x) = \int_a^b \sum_{n=0}^{N} c_n (x-s)^n f(s)\,ds$$

Bring the sum and c_n in front of the integral.

$$g(x) = \sum_{n=0}^{N} c_n \int_a^b (x-s)^n f(s)\,ds$$

Use the binomial theorem here to separate the x and s variables.

$$g(x) = \sum_{n=0}^{N} c_n \int_a^b \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k (-s)^{n-k} f(s)\,ds$$

Bring the sum and the constants in front of the integral.

$$g(x) = \sum_{n=0}^{N} \sum_{k=0}^{n} c_n \frac{n!}{(n-k)!k!} x^k \int_a^b (-s)^{n-k} f(s)\,ds$$

Interchange the order of summation—it works the same way as if the sums were integrals, but rather than a continuous region we have a grid here in the kn-plane.

$$g(x) = \sum_{k=0}^{N} \sum_{n=k}^{N} c_n \frac{n!}{(n-k)!k!} x^k \int_a^b (-s)^{n-k} f(s)\,ds$$

c_n, the factorials, and the definite integral are all constants. Group them in front of x^k.

$$g(x) = \sum_{k=0}^{N} \left[\sum_{n=k}^{N} c_n \frac{n!}{(n-k)!k!} \int_a^b (-s)^{n-k} f(s)\,ds \right] x^k$$

The term in square brackets can be written compactly as d_k.

$$g(x) = \sum_{k=0}^{N} d_k x^k$$

Therefore, $g(x)$ is also a polynomial.