Exercise 5

In the three-dimensional half-space $\{(x, y, z) : z > 0\}$, solve the Laplace equation with $u(x, y, 0) = \delta(x, y)$, where δ denotes the delta function, as follows.

(a) Show that

$$u(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikx+ily} e^{-z\sqrt{k^2+l^2}} \frac{dk\,dl}{4\pi^2}.$$

(b) Letting $\rho = \sqrt{k^2 + l^2}$, $r = \sqrt{x^2 + y^2}$, and θ be the angle between (x, y) and (k, l), so that $xk + yl = \rho r \cos\theta$, show that

$$u(x, y, z) = \int_0^{2\pi} \int_0^{\infty} e^{i\rho r \cos\theta} e^{-z\rho} \rho\, d\rho \frac{d\theta}{4\pi^2}.$$

(c) Carry out the integral with respect to ρ and then use an extensive table of integrals to evaluate the θ integral.