Exercise 1

Use the Fourier transform directly to solve the heat equation with a convection term, namely, 
\[ u_t = \kappa u_{xx} + \mu u_x \] for \(-\infty < x < \infty\), with an initial condition \(u(x, 0) = \phi(x)\), assuming that \(u(x, t)\) is bounded and \(\kappa > 0\).

Solution

Since the PDE is linear and the \(x\) variable goes from \(-\infty\) to \(\infty\), the Fourier transform can be applied to solve it. Here we define the Fourier transform of a function \(u(x, t)\) as
\[
\mathcal{F}\{u(x, t)\} = U(\omega, t) = \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} \, dx.
\]

(\(\omega\) is used rather than \(k\) to avoid confusion with \(\kappa\) in the PDE.) As a result, the derivatives of \(u\) with respect to \(x\) and \(t\) transform as follows.
\[
\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (i\omega)^n U(\omega, t) \\
\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}
\]

Take the Fourier transform of both sides of the PDE
\[
\mathcal{F}\{u_t\} = \mathcal{F}\{\kappa u_{xx} + \mu u_x\}
\]
and its initial condition.
\[
\mathcal{F}\{u(x, 0)\} = \mathcal{F}\{\phi(x)\} \quad \rightarrow \quad U(\omega, 0) = \Phi(\omega)
\] (1)

Use the fact that the Fourier transform is a linear operator.
\[
\mathcal{F}\{u_t\} = \kappa \mathcal{F}\{u_{xx}\} + \mu \mathcal{F}\{u_x\}
\]

Transform the derivatives with the expressions above.
\[
\frac{dU}{dt} = \kappa(i\omega)^2 U(\omega, t) + \mu(i\omega) U(\omega, t)
\]

The second-order PDE has thus been reduced to a first-order ODE that can be solved with separation of variables.
\[
\frac{dU}{dt} = (-\kappa \omega^2 + i\mu \omega) U
\]

Separate variables.
\[
\frac{dU}{U} = (-\kappa \omega^2 + i\mu \omega) \, dt
\]

Integrate both sides.
\[
\ln |U| = (-\kappa \omega^2 + i\mu \omega) t + C
\]

Exponentiate both sides.
\[
|U| = e^C e^{-\kappa \omega^2 + i\mu \omega} t
\]
Remove the absolute value sign by introducing ± on the right side.

\[ U(\omega, t) = \pm e^{Ct}e^{(-\kappa\omega^2 + i\mu\omega)t} \]

Use a new constant of integration.

\[ U(\omega, t) = A(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t} \]

Use the Fourier-transformed initial condition in equation (1) to determine \( A(\omega) \).

\[ U(\omega, 0) = A(\omega) = \Phi(\omega) \]

So we have

\[ U(\omega, t) = \Phi(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t}. \]

Now that we have \( U(\omega, t) \), we can change back to \( u(x, t) \) by taking the inverse Fourier transform of it.

\[
\begin{align*}
    u(x, t) &= \mathcal{F}^{-1}\{U(\omega, t)\} \\
    &= \mathcal{F}^{-1}\{A(\omega)e^{(-\kappa\omega^2 + i\mu\omega)t}\}
\end{align*}
\]

Because we are taking the inverse Fourier transform of a product of two functions, the convolution theorem can be applied, which states

\[
\mathcal{F}^{-1}\{\Phi(\omega)G(\omega)\} = \int_{-\infty}^{\infty} \phi(x-s)g(s) \, ds = \int_{-\infty}^{\infty} \phi(s)g(x-s) \, ds.
\]

All that we have to do then is calculate \( g \), the inverse Fourier transform of the exponential function, and we can write the solution using the convolution theorem.

\[
\mathcal{F}^{-1}\{e^{(-\kappa\omega^2 + i\mu\omega)t}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(-\kappa\omega^2 + i\mu\omega)t} e^{i\omega x} \, d\omega
\]

Combine the exponential functions and then proceed to complete the square in the exponent.

\[
\begin{align*}
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[(-\kappa\omega^2 + i\mu\omega)t + i\omega x\right] \, d\omega \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[-\kappa t\omega^2 + i(\mu t + x)\omega\right] \, d\omega \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega^2 - \frac{i(\mu t + x)}{\kappa t}\omega\right]\right\} \, d\omega \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega^2 - \frac{i(\mu t + x)}{\kappa t}\omega + \frac{i^2(\mu t + x)^2}{4\kappa^2 t^2}\right] + \kappa t\frac{(\mu t + x)^2}{4\kappa^2 t^2}\right\} \, d\omega \\
    &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega - \frac{i(\mu t + x)}{2\kappa t}\right]^2 - \frac{(\mu t + x)^2}{4\kappa t}\right\} \, d\omega \\
    &= \frac{1}{2\pi} \exp\left[-\frac{(\mu t + x)^2}{4\kappa t}\right] \int_{-\infty}^{\infty} \exp\left\{-\kappa t\left[\omega - \frac{i(\mu t + x)}{2\kappa t}\right]^2\right\} \, d\omega
\end{align*}
\]
Make the following substitution in the integral.

\[ p = \omega - \frac{i(\mu t + x)}{2\kappa t} \]
\[ dp = d\omega \]

We obtain

\[ \mathcal{F}^{-1}\{e^{-\kappa \omega^2 + i\mu \omega}t\} = \frac{1}{2\pi} \exp \left[ -\frac{(\mu t + x)^2}{4\kappa t} \right] \int_{-\infty}^{\infty} e^{-\kappa r^2} dp. \]

Make another substitution.

\[ r = \sqrt{\kappa t} p \quad \rightarrow \quad r^2 = \kappa t p^2 \]
\[ dr = \sqrt{\kappa t} dp \quad \rightarrow \quad \frac{1}{\sqrt{\kappa t}} dr = dp \]

We obtain

\[ \mathcal{F}^{-1}\{e^{-\kappa \omega^2 + i\mu \omega}t\} = \frac{1}{2\pi} \exp \left[ -\frac{(\mu t + x)^2}{4\kappa t} \right] \int_{-\infty}^{\infty} e^{-r^2} \left( \frac{1}{\sqrt{\kappa t}} dr \right) \]
\[ = \frac{1}{2\pi \sqrt{\kappa t}} \exp \left[ -\frac{(\mu t + x)^2}{4\kappa t} \right] \int_{-\infty}^{\infty} e^{-r^2} dr \]
\[ = \frac{1}{2\pi \sqrt{\kappa t}} \exp \left[ -\frac{(\mu t + x)^2}{4\kappa t} \right] \cdot \sqrt{\pi} \]
\[ = \frac{1}{2\sqrt{\pi \kappa t}} \exp \left[ -\frac{(\mu t + x)^2}{4\kappa t} \right]. \]

By the convolution theorem then,

\[ u(x, t) = \int_{-\infty}^{\infty} \phi(s) \frac{1}{2\sqrt{\pi \kappa t}} \exp \left[ -\frac{(\mu t + x - s)^2}{4\kappa t} \right] ds. \]

Therefore,

\[ u(x, t) = \frac{1}{\sqrt{4\pi \kappa t}} \int_{-\infty}^{\infty} \phi(s) \exp \left[ -\frac{(\mu t + x - s)^2}{4\kappa t} \right] ds. \]