Exercise 2

Use the Fourier transform in the x variable to find the harmonic function in the half-plane $\{y > 0\}$ that satisfies the Neumann condition $\partial u/\partial y = h(x)$ on $\{y = 0\}$.

Solution

A harmonic function is a function that satisfies the Laplace equation. The boundary value problem we have to solve then is

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \quad \frac{\partial u}{\partial y}(x, 0) = h(x)$$

with the additional requirement that u remains bounded as $y \to \infty$.

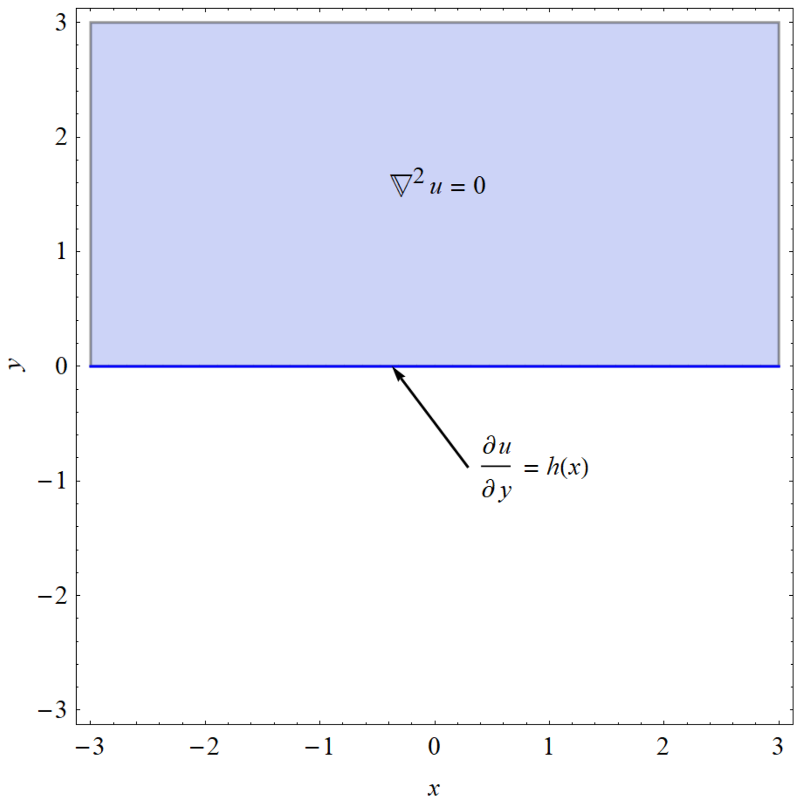

Figure 1: This is an illustration of the problem. The Laplace equation is being solved in the upper half of the xy-plane with the derivative of u at the $y = 0$ boundary specified to be $h(x)$.

Since the PDE is linear and the x variable goes from $-\infty$ to ∞, the Fourier transform can be applied to solve it. Here we define the Fourier transform of a function $u(x, y)$ as

$$\mathcal{F}\{u(x, y)\} = U(k, y) = \int_{-\infty}^{\infty} u(x, y)e^{-ikx}\, dx.$$

As a result, the derivatives of u with respect to x and y transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,y)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = \frac{d^n U}{dy^n}$$

Take the Fourier transform of both sides of the PDE

$$\mathcal{F}\{u_{xx} + u_{yy}\} = \mathcal{F}\{0\}$$

and the boundary condition.

$$\mathcal{F}\left\{\frac{\partial u}{\partial y}(x,0)\right\} = \mathcal{F}\{h(x)\} \quad \rightarrow \quad \frac{dU}{dy}(k,0) = H(k) \tag{1}$$

Use the fact that the Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = 0$$

Transform the derivatives with the expressions above.

$$(ik)^2 U(k,y) + \frac{d^2 U}{dy^2} = 0$$

The second-order PDE has thus been reduced to a second-order ODE whose solution can be written in terms of exponential functions.

$$\frac{d^2 U}{dy^2} = k^2 U$$

$$U(k,y) = A(k)e^{ky} + B(k)e^{-ky}$$

Because the solution has to remain bounded as $y \to \infty$, we conclude that $A(k) = 0$ when $k > 0$ and $B(k) = 0$ when $k < 0$. The absolute value sign can be used to represent the formula compactly.

$$U(k,y) = C(k)e^{-|k|y}$$

Differentiate this function with respect to y.

$$\frac{dU}{dy}(k,y) = -|k|C(k)e^{-|k|y}$$

Apply the Fourier-transformed boundary condition in equation (1) to determine $C(k)$.

$$\frac{dU}{dy}(k,0) = -|k|C(k) = H(k) \quad \rightarrow \quad C(k) = -\frac{H(k)}{|k|}$$

Thus,

$$U(k,y) = -\frac{H(k)}{|k|}e^{-|k|y}.$$

Now that we have $U(k, y)$, we can change back to $u(x, y)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x,y) &= \mathcal{F}^{-1}\{U(k,y)\} \\ &= \mathcal{F}^{-1}\left\{-\frac{H(k)}{|k|}e^{-|k|y}\right\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, the convolution theorem can be applied, which states

$$\mathcal{F}^{-1}\{H(k)G(k)\} = \int_{-\infty}^{\infty} h(x-s)g(s)\,ds = \int_{-\infty}^{\infty} h(s)g(x-s)\,ds.$$

All that we have to do then is calculate g, the inverse Fourier transform of $-e^{-|k|y}/|k|$.

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{-e^{-|k|y}}{|k|}\right\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-e^{-|k|y}}{|k|}e^{ikx}\,dk \\ &= -\frac{1}{2\pi}[\ln(x^2+y^2)+2\gamma], \end{aligned}$$

where γ is the Euler constant defined by

$$\gamma = -\int_0^{\infty} e^{-k}\ln k\,dk \approx 0.577.$$

By the convolution theorem then,

$$u(x,y) = \int_{-\infty}^{\infty} h(s)\left\{-\frac{1}{2\pi}\{\ln[(x-s)^2+y^2]+2\gamma\}\right\} ds.$$

Therefore,

$$u(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} h(s)\{\ln[(x-s)^2+y^2]+2\gamma\}\,ds.$$