Exercise 2

Use the Fourier transform in the $x$ variable to find the harmonic function in the half-plane \( \{ y > 0 \} \) that satisfies the Neumann condition $\partial u / \partial y = h(x)$ on $\{ y = 0 \}$.

Solution

A harmonic function is a function that satisfies the Laplace equation. The boundary value problem we have to solve then is

\[
\nabla^2 u = u_{xx} + u_{yy} = 0, \quad \frac{\partial u}{\partial y}(x, 0) = h(x)
\]

with the additional requirement that $u$ remains bounded as $y \to \infty$.

Figure 1: This is an illustration of the problem. The Laplace equation is being solved in the upper half of the $xy$-plane with the derivative of $u$ at the $y = 0$ boundary specified to be $h(x)$.

Since the PDE is linear and the $x$ variable goes from $-\infty$ to $\infty$, the Fourier transform can be applied to solve it. Here we define the Fourier transform of a function $u(x, y)$ as

\[
\mathcal{F}\{u(x, y)\} = U(k, y) = \int_{-\infty}^{\infty} u(x, y)e^{-ikx} \, dx.
\]
As a result, the derivatives of \( u \) with respect to \( x \) and \( y \) transform as follows.

\[
\mathcal{F}\left\{ \frac{\partial^n u}{\partial x^n} \right\} = (ik)^n U(k, y)
\]

\[
\mathcal{F}\left\{ \frac{\partial^n u}{\partial y^n} \right\} = \frac{d^n U}{dy^n}
\]

Take the Fourier transform of both sides of the PDE

\[
\mathcal{F}\{u_{xx} + u_{yy}\} = \mathcal{F}\{0\}
\]

and the boundary condition.

\[
\mathcal{F}\left\{ \frac{\partial u}{\partial y}(x, 0) \right\} = \mathcal{F}\{h(x)\} \rightarrow \frac{dU}{dy}(k, 0) = H(k) \quad (1)
\]

Use the fact that the Fourier transform is a linear operator.

\[
\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} = 0
\]

Transform the derivatives with the expressions above.

\[
(ik)^2 U(k, y) + \frac{d^2 U}{dy^2} = 0
\]

The second-order PDE has thus been reduced to a second-order ODE whose solution can be written in terms of exponential functions.

\[
\frac{d^2 U}{dy^2} = k^2 U
\]

\[
U(k, y) = A(k)e^{ky} + B(k)e^{-ky}
\]

Because the solution has to remain bounded as \( y \to \infty \), we conclude that \( A(k) = 0 \) when \( k > 0 \) and \( B(k) = 0 \) when \( k < 0 \). The absolute value sign can be used to represent the formula compactly.

\[
U(k, y) = C(k)e^{-|k|y}
\]

Differentiate this function with respect to \( y \).

\[
\frac{dU}{dy}(k, y) = -|k|C(k)e^{-|k|y}
\]

Apply the Fourier-transformed boundary condition in equation (1) to determine \( C(k) \).

\[
\frac{dU}{dy}(k, 0) = -|k|C(k) = H(k) \quad \rightarrow \quad C(k) = -\frac{H(k)}{|k|}
\]

Thus,

\[
U(k, y) = -\frac{H(k)}{|k|}e^{-|k|y}
\]

www.stemjock.com
Now that we have $U(k, y)$, we can change back to $u(x, y)$ by taking the inverse Fourier transform of it.

\[
    u(x, y) = \mathcal{F}^{-1}\{U(k, y)\} \\
    = \mathcal{F}^{-1}\left\{-\frac{H(k)}{|k|} e^{-|k|y}\right\}
\]

Because we are taking the inverse Fourier transform of a product of two functions, the convolution theorem can be applied, which states

\[
    \mathcal{F}^{-1}\{H(k)G(k)\} = \int_{-\infty}^{\infty} h(x - s)g(s) \, ds = \int_{-\infty}^{\infty} h(s)g(x - s) \, ds.
\]

All that we have to do then is calculate $g$, the inverse Fourier transform of $-\frac{e^{-|k|y}}{|k|}$.

\[
    \mathcal{F}^{-1}\left\{-\frac{e^{-|k|y}}{|k|}\right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-e^{-|k|y}}{|k|} e^{ikx} \, dk \\
    = -\frac{1}{2\pi} \left[ \ln(x^2 + y^2) + 2\gamma \right],
\]

where $\gamma$ is the Euler constant defined by

\[
    \gamma = -\int_{0}^{\infty} e^{-k} \ln k \, dk \approx 0.577.
\]

By the convolution theorem then,

\[
    u(x, y) = \int_{-\infty}^{\infty} h(s) \left\{-\frac{1}{2\pi} \left[ \ln((x - s)^2 + y^2) + 2\gamma \right] \right\} ds.
\]

Therefore,

\[
    u(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} h(s) \{\ln((x - s)^2 + y^2) + 2\gamma\} \, ds.
\]