Exercise 3

Use the Fourier transform to find the bounded solution of the equation $-\Delta u + m^2 u = \delta(x)$ in free three-dimensional space with $m > 0$.

Solution

$\Delta$ is the Laplacian operator ($\nabla^2$), so the PDE to solve here is

$$-(u_{xx} + u_{yy} + u_{zz}) + m^2 u = \delta(x, y, z), \quad -\infty < x, y, z < \infty.$$ 

Since the PDE is linear and the $x$, $y$, and $z$ variables go from $-\infty$ to $\infty$, the triple Fourier transform can be applied to solve it. Here we define the triple Fourier transform of a function $u(x, y, z)$ as

$$\mathcal{F}\{u(x, y, z)\} = U(k, l, M) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y, z)e^{-i(kx+ly+Mz)} \, dx \, dy \, dz.$$ 

As a result, the derivatives of $u$ with respect to $x$ and $y$ and $z$ transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k, l, M)$$

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = (il)^n U(k, l, M)$$

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial z^n}\right\} = (iM)^n U(k, l, M)$$

Take the triple Fourier transform of both sides of the PDE.

$$\mathcal{F}\{-u_{xx} + u_{yy} + u_{zz}\} + m^2 \mathcal{F}\{u\} = \mathcal{F}\{\delta(x, y, z)\}$$

Use the fact that the operator is linear on the left side and apply the definition on the right side.

$$-\mathcal{F}\{u_{xx}\} - \mathcal{F}\{u_{yy}\} - \mathcal{F}\{u_{zz}\} + m^2 \mathcal{F}\{u\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y, z)e^{-i(kx+ly+Mz)} \, dx \, dy \, dz$$

Transform the derivatives with the expressions above.

$$-(ik)^2 U(k, l, M) - (il)^2 U(k, l, M) - (iM)^2 U(k, l, M) + m^2 U(k, l, M) = e^{-i(k\cdot0+l\cdot0+M\cdot0)}$$

Factor the left side.

$$(k^2 + l^2 + M^2 + m^2)U(k, l, M) = 1$$

Thus,

$$U(k, l, M) = \frac{1}{k^2 + l^2 + M^2 + m^2}.$$ 

Now that we have $U(k, l, M)$, we can change back to $u(x, y, z)$ by taking the inverse triple Fourier transform of it.

$$u(x, y, z) = \mathcal{F}^{-1}\{U(k, l, M)\} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k^2 + l^2 + M^2 + m^2} e^{i(kx+ly+Mz)} \, dk \, dl \, dM$$

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In order to solve this triple integral, we will switch to spherical coordinates with \( \rho^2 = k^2 + l^2 + M^2 \) and the polar axis pointing in the direction of \( \mathbf{r} = (x, y, z) \) so that \( \mathbf{\rho} \cdot \mathbf{r} = kx + ly + Mz = pr \cos \phi \).

\[
\begin{align*}
  u(x, y, z) &= \frac{1}{(2\pi)^3} \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{1}{\rho^2 + m^2} e^{i\rho r \cos \phi} \rho^2 \sin \phi \, d\rho \
  &= \frac{1}{8\pi^3} \left( \int_0^{2\pi} d\theta \right) \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left( \int_0^\pi e^{i\rho r \cos \phi} \sin \phi \, d\phi \right) \, d\rho \\
  &= \frac{1}{4\pi^2} \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left( \int_0^\pi e^{i\rho r \cos \phi} \sin \phi \, d\phi \right) \, d\rho 
\end{align*}
\]

Use a substitution to solve the integral in parentheses.

\[
s = i\rho r \cos \phi \quad \Rightarrow \quad ds = i\rho r (-\sin \phi) \, d\phi 
\]

Consequently,

\[
\begin{align*}
  u(x, y, z) &= \frac{1}{4\pi^2} \int_0^\infty \frac{\rho^2}{\rho^2 + m^2} \left[ \int_{-i\rho r}^{i\rho r} e^s \left( -\frac{ds}{i\rho r} \right) \right] \, d\rho \\
  &= \frac{1}{4\pi^2r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} \left( \int_{-i\rho r}^{i\rho r} \frac{e^s}{i} \right) \, d\rho \\
  &= \frac{1}{4\pi^2r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} \left( \frac{e^{i\rho r} - e^{-i\rho r}}{i} \right) \, d\rho \\
  &= \frac{1}{4\pi^2r} \int_0^\infty \frac{\rho}{\rho^2 + m^2} (2 \sin \rho r) \, d\rho.
\end{align*}
\]

The integrand is even in \( \rho \), so the factor of 2 can be used to extend the interval of integration to \(( -\infty, \infty )\).

\[
\begin{align*}
  u(x, y, z) &= \frac{1}{4\pi^2r} \int_{-\infty}^{\infty} \frac{\rho}{\rho^2 + m^2} \sin \rho r \, d\rho \\
  \text{The theory of residues will be used to evaluate the final integral. Consider the function,}
\end{align*}
\]

\[
f(z) = \frac{z}{z^2 + m^2} = \frac{z}{(z + im)(z - im)},
\]

where \( z \) is a complex number. \( f(z)e^{irz} \) is analytic everywhere in the complex plane except at the singularities, \( z = -im \) and \( z = im \). \( z = im \) lies inside the semicircular region shown in Figure 1.
Integrating $f(z)e^{irz}$ around the boundary $C$ of the semicircle in Figure 1 gives us

$$\int_{C} f(z)e^{irz} \, dz = \int_{-R}^{R} f(z)e^{irz} \, dz + \int_{C_R} f(z)e^{irz} \, dz,$$

which is equal to $2\pi i$ times the residue at $z = im$ according to Cauchy’s residue theorem.

$$\int_{-R}^{R} f(z)e^{irz} \, dz + \int_{C_R} f(z)e^{irz} \, dz = 2\pi i \text{Res}_{z=im} f(z)e^{irz}$$

Because $z = im$ is a simple pole, we have

$$\text{Res}_{z=im} f(z)e^{irz} = \phi(im), \quad \text{where} \quad \phi(z) = \frac{z}{(z + im)}e^{irz}.$$

As a result,

$$\int_{-R}^{R} f(z)e^{irz} \, dz + \int_{C_R} f(z)e^{irz} \, dz = 2\pi i \cdot \frac{im}{2im} e^{-mr} = i\pi e^{-mr}.$$  

Now take the limit as $R \to \infty$.

$$\int_{-\infty}^{\infty} f(z)e^{irz} \, dz + \lim_{R\to\infty} \int_{C_R} f(z)e^{irz} \, dz = i\pi e^{-mr}$$

The limit of the second integral is equal to zero by Jordan’s lemma because

$$\left| \int_{C_R} f(z)e^{irz} \, dz \right| \leq \int_{C_R} |f(z)e^{irz}| \, dz \leq \pi R \cdot \frac{R}{R^2 + m^2} \quad \text{and} \quad \lim_{R \to \infty} \frac{R}{R^2 + m^2} = 0.$$

What remains then is

$$\int_{-\infty}^{\infty} f(z)e^{irz} \, dz = i\pi e^{-mr}.$$  

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Use Euler’s formula here.

\[ \int_{-\infty}^{\infty} f(z)(\cos rz + i \sin rz) \, dz = i\pi e^{-mr} \]

Write the integral as the sum of a real part and an imaginary part.

\[ \int_{-\infty}^{\infty} f(z) \cos rz \, dz + i \int_{-\infty}^{\infty} f(z) \sin rz \, dz = i\pi e^{-mr} \]

From this equation, by matching the real and imaginary parts on both sides, we deduce the following integral formulas.

\[ \int_{-\infty}^{\infty} \frac{z}{z^2 + m^2} \cos rz \, dz = 0 \]

\[ \int_{-\infty}^{\infty} \frac{z}{z^2 + m^2} \sin rz \, dz = \pi e^{-mr} \]

Use the second formula in the equation for \( u(x, y, z) \).

\[
\begin{align*}
    u(x, y, z) &= \frac{1}{4\pi^2r} \int_{-\infty}^{\infty} \frac{\rho}{\rho^2 + m^2} \sin \rho r \, d\rho \\
               &= \frac{1}{4\pi^2r} \cdot \pi e^{-mr} \\
               &= \frac{1}{4\pi r} e^{-mr}
\end{align*}
\]

Therefore,

\[ u(x, y, z) = \frac{e^{-m\sqrt{x^2+y^2+z^2}}}{4\pi \sqrt{x^2 + y^2 + z^2}}. \]