Exercise 5

In the three-dimensional half-space $\{(x, y, z) : z > 0\}$, solve the Laplace equation with $u(x, y, 0) = \delta(x, y)$, where δ denotes the delta function, as follows.

(a) Show that

$$u(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ikx+ily}e^{-z\sqrt{k^2+l^2}}\,\frac{dk\,dl}{4\pi^2}.$$

(b) Letting $\rho = \sqrt{k^2 + l^2}$, $r = \sqrt{x^2 + y^2}$, and θ be the angle between (x, y) and (k, l), so that $xk + yl = \rho r\cos\theta$, show that

$$u(x, y, z) = \int_0^{2\pi}\int_0^{\infty} e^{i\rho r\cos\theta}e^{-z\rho}\rho\,d\rho\frac{d\theta}{4\pi^2}.$$

(c) Carry out the integral with respect to ρ and then use an extensive table of integrals to evaluate the θ integral.

Solution

Part (a)

The Laplace equation in the upper half-space is

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0, \quad -\infty < x, y < \infty,\ z > 0$$

subject to the boundary condition,

$$u(x, y, 0) = \delta(x, y),$$

and the requirement that u remains bounded as $z \to \infty$. Since the PDE is linear and the x and y variables go from $-\infty$ to ∞, the double Fourier transform can be applied to solve it. Here we define the double Fourier transform of a function $u(x, y, z)$ as

$$\mathcal{F}\{u(x, y, z)\} = U(k, l, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x, y, z)e^{-i(kx+ly)}\,dx\,dy.$$

As a result, the derivatives of u with respect to x and y and z transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k, l, z)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} = (il)^n U(k, l, z)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n U}{dz^n}$$

Take the double Fourier transform of both sides of the PDE

$$\mathcal{F}\{u_{xx} + u_{yy} + u_{zz}\} = \mathcal{F}\{0\}$$

and the boundary condition.

$$\begin{aligned} u(x,y,0) = \delta(x,y) \quad \rightarrow \quad \mathcal{F}\{u(x,y,0)\} &= \mathcal{F}\{\delta(x,y)\} \\ U(k,l,0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)e^{-i(kx+ly)}\,dx\,dy \\ &= e^{-i(k\cdot 0 + l\cdot 0)} \\ &= 1 \end{aligned} \tag{1}$$

Use the fact that the double Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{yy}\} + \mathcal{F}\{u_{zz}\} = 0$$

Transform the derivatives with the expressions above.

$$(ik)^2U(k,l,z) + (il)^2U(k,l,z) + \frac{d^2U}{dz^2} = 0$$

$$-(k^2+l^2)U + \frac{d^2U}{dz^2} = 0$$

$$\frac{d^2U}{dz^2} = (k^2+l^2)U$$

The second-order PDE has thus been reduced to a second-order ODE whose solution can be written in terms of exponential functions.

$$U(k,l,z) = A(k,l)e^{z\sqrt{k^2+l^2}} + B(k,l)e^{-z\sqrt{k^2+l^2}}$$

For u to remain bounded as $z \to \infty$, we conclude that $A(k,l) = 0$.

$$U(k,l,z) = B(k,l)e^{-z\sqrt{k^2+l^2}}$$

Apply the Fourier-transformed boundary condition in equation (1) here to determine $B(k,l)$.

$$U(k,l,0) = B(k,l) = 1$$

Thus,

$$U(k,l,z) = e^{-z\sqrt{k^2+l^2}}.$$

Now that we have $U(k,l,z)$, we can change back to $u(x,y,z)$ by taking the inverse double Fourier transform of it.

$$\begin{aligned} u(x,y,z) &= \mathcal{F}^{-1}\{U(k,l,z)\} \\ &= \mathcal{F}^{-1}\left\{e^{-z\sqrt{k^2+l^2}}\right\} \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-z\sqrt{k^2+l^2}}e^{i(kx+ly)}\,dk\,dl \end{aligned}$$

Therefore,

$$u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ikx+ily}e^{-z\sqrt{k^2+l^2}}\,\frac{dk\,dl}{4\pi^2}.$$

Part (b)

In order to solve the double integral, we will make the prescribed substitutions,

$$\begin{aligned}\rho &= \sqrt{k^2+l^2}\\ r\rho\cos\theta &= kx+ly,\end{aligned}$$

where $r = \sqrt{x^2+y^2}$. There are two equations, so we can solve for k and l in terms of the new variables, ρ and θ.

$$\begin{aligned}k &= \frac{\rho}{r}(x\cos\theta + y\sin\theta)\\ l &= \frac{\rho}{r}(y\cos\theta - x\sin\theta)\end{aligned}$$

Now we can calculate the Jacobian.

$$\begin{aligned}\frac{\partial(k,l)}{\partial(\rho,\theta)} &= \begin{vmatrix}\frac{\partial k}{\partial\rho} & \frac{\partial k}{\partial\theta}\\ \frac{\partial l}{\partial\rho} & \frac{\partial l}{\partial\theta}\end{vmatrix}\\ &= \begin{vmatrix}\frac{x\cos\theta+y\sin\theta}{r} & \frac{\rho}{r}(y\cos\theta - x\sin\theta)\\ \frac{y\cos\theta-x\sin\theta}{r} & -\frac{\rho}{r}(x\cos\theta+y\sin\theta)\end{vmatrix}\\ &= -\frac{\rho}{r^2}(x\cos\theta+y\sin\theta)^2 - \frac{\rho}{r^2}(y\cos\theta-x\sin\theta)^2\\ &= -\frac{\rho}{r^2}[x^2(\sin^2\theta+\cos^2\theta)+y^2(\sin^2\theta+\cos^2\theta)]\\ &= -\rho\end{aligned}$$

Consequently, the double integral becomes

$$\begin{aligned}u(x,y,z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ikx+ily}e^{-z\sqrt{k^2+l^2}}\,\frac{dk\,dl}{4\pi^2}\\ &= \iint_R e^{ir\rho\cos\theta}e^{-z\rho}\left|\frac{\partial(k,l)}{\partial(\rho,\theta)}\right|\frac{d\rho\,d\theta}{4\pi^2},\end{aligned}$$

where R is the domain of integration in terms of the new coordinates. ρ is the distance from the origin in the kl-plane, so it extends from 0 to ∞. θ is the angle from the vector $\langle x, y\rangle$ to the vector $\langle k, l\rangle$, so it extends from 0 to 2π.

$$= \int_0^{2\pi}\int_0^{\infty} e^{ir\rho\cos\theta}e^{-z\rho}|-\rho|\,\frac{d\rho\,d\theta}{4\pi^2}$$

Therefore,

$$u(x,y,z) = \int_0^{2\pi}\int_0^{\infty} e^{i\rho r\cos\theta}e^{-z\rho}\rho\,d\rho\frac{d\theta}{4\pi^2}.$$

Part (c)

Combine the exponential functions, factor ρ in the exponent, and bring the constant in front of the integral.

$$u(x,y,z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} e^{\rho(ir\cos\theta - z)}\rho\,d\rho\,d\theta$$

Use integration by parts to do the inner integral.

$$\begin{aligned} u(x,y,z) &= \frac{1}{4\pi^2}\int_0^{2\pi}\left[\frac{\rho}{ir\cos\theta - z}e^{\rho(ir\cos\theta-z)} - \frac{1}{(ir\cos\theta - z)^2}e^{\rho(ir\cos\theta-z)}\right]\Bigg|_0^{\infty} d\theta \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\frac{1}{(ir\cos\theta - z)^2}\,d\theta \end{aligned}$$

The integral we need from a table is

$$\int_0^{2\pi}\frac{1}{(a\cos\theta + b)^2}\,d\theta = \frac{2\pi b}{(a-b)^2(a+b)}\sqrt{1-\frac{2a}{a+b}},$$

where a is assumed to be complex and b is assumed to be real. Plugging in ir for a and $-z$ for b, we have

$$\begin{aligned} u(x,y,z) &= \frac{1}{4\pi^2}\cdot\frac{2\pi(-z)}{(ir+z)^2(ir-z)}\sqrt{1-\frac{2ir}{ir-z}} \\ &= \frac{1}{2\pi}\cdot\frac{(-z)}{(ir+z)^2(ir-z)}\sqrt{\frac{ir-z-2ir}{ir-z}} \\ &= \frac{1}{2\pi}\cdot\frac{(-z)}{(ir+z)^2(ir-z)}\sqrt{\frac{-ir-z}{ir-z}} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(ir+z)^2(ir-z)}\sqrt{\frac{ir+z}{ir-z}} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(ir+z)^{3/2}(ir-z)^{3/2}} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(-r^2-z^2)^{3/2}} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(\sqrt{-r^2-z^2})^3} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(i\sqrt{r^2+z^2})^3} \\ &= \frac{1}{2\pi}\cdot\frac{(-i)z}{(-i)(\sqrt{r^2+z^2})^3}. \end{aligned}$$

Therefore,

$$u(x,y,z) = \frac{1}{2\pi}\cdot\frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

This answer is in disagreement with the answer at the back of the book [$u = 2\pi z/(r^2+z^2)^{3/2}$]. I believe Mr. Strauss forgot about the factor of $1/4\pi^2$.