Exercise 2

Verify each entry in the table of properties of the Laplace transform.

Solution

The table of properties of the Laplace transform the exercise is referring to is given below.

	Function	**Transform**
(i)	$af(t) + bg(t)$	$aF(s) + bG(s)$
(ii)	$\frac{df}{dt}$	$sF(s) - f(0)$
(iii)	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$
(iv)	$e^{bt} f(t)$	$F(s - b)$
(v)	$\frac{f(t)}{t}$	$\int_s^\infty F(s')\, ds'$
(vi)	$tf(t)$	$-\frac{dF}{ds}$
(vii)	$H(t - b)f(t - b)$	$e^{-bs} F(s)$
(viii)	$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$
(ix)	$\int_0^t g(t - t')f(t')\, dt'$	$F(s)G(s)$

We will verify these properties using the definition of the Laplace transform one at a time. The Laplace transform of a function, $f(t)$, is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\, dt.$$

Proof of Property I

Taking the Laplace transform of the left side of property (i):

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty [af(t) + bg(t)]\, e^{-st}\, dt \\
&= \int_0^\infty \left[af(t)e^{-st} + bg(t)e^{-st}\right]\, dt \\
&= \int_0^\infty af(t)e^{-st}\, dt + \int_0^\infty bg(t)e^{-st}\, dt \\
&= a\int_0^\infty f(t)e^{-st}\, dt + b\int_0^\infty g(t)e^{-st}\, dt \\
&= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \\
&= aF(s) + bG(s)
\end{aligned}$$

The first property is verified. Moving on to property (ii).

Proof of Property II

$$\begin{aligned}
\mathcal{L}\left\{\frac{df}{dt}\right\} &= \int_0^\infty \frac{df}{dt}e^{-st}\,dt \\
&= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (-s)f(t)e^{-st}\,dt \\
&= \underbrace{\lim_{t\to\infty} f(t)e^{-st}}_{=0} - f(0) + s\int_0^\infty f(t)e^{-st}\,dt \\
&= -f(0) + s\mathcal{L}\{f(t)\} \\
&= sF(s) - f(0)
\end{aligned}$$

The second property is verified. Moving on to property (iii).

Proof of Property III

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} &= \int_0^\infty \frac{d^2f}{dt^2}e^{-st}\,dt \\
&= \frac{df}{dt}e^{-st}\Big|_0^\infty - \int_0^\infty (-s)\frac{df}{dt}e^{-st}\,dt \\
&= \underbrace{\lim_{t\to\infty} \frac{df}{dt}e^{-st}}_{=0} - \frac{df}{dt}\Big|_{t=0} + s\int_0^\infty \frac{df}{dt}e^{-st}\,dt \\
&= -f'(0) + s\left[f(t)e^{-st}\Big|_0^\infty - \int_0^\infty (-s)f(t)e^{-st}\,dt\right] \\
&= -f'(0) + s\underbrace{\lim_{t\to\infty} f(t)e^{-st}}_{=0} - sf(0) + s^2\int_0^\infty f(t)e^{-st}\,dt \\
&= -f'(0) - sf(0) + s^2\mathcal{L}\{f(t)\} \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

The third property is verified. Moving on to property (iv).

Proof of Property IV

$$\begin{aligned}
\mathcal{L}\{e^{bt}f(t)\} &= \int_0^\infty \left[e^{bt}f(t)\right]e^{-st}\,dt \\
&= \int_0^\infty e^{bt}f(t)e^{-st}\,dt \\
&= \int_0^\infty f(t)e^{bt-st}\,dt \\
&= \int_0^\infty f(t)e^{-(s-b)t}\,dt \\
&= F(s-b)
\end{aligned}$$

The fourth property is verified. Moving on to property (v).

Proof of Property V

We have to show that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s')\,ds'.$$

Working with the right side:

$$\begin{aligned}
\int_s^\infty F(s')\,ds' &= \int_s^\infty \int_0^\infty f(t)e^{-s't}\,dt\,ds' \\
&= \int_0^\infty \int_s^\infty f(t)e^{-s't}\,ds'\,dt \\
&= \int_0^\infty f(t)\left[\int_s^\infty e^{-s't}\,ds'\right]dt \\
&= \int_0^\infty f(t)\left(\frac{1}{-t}e^{-s't}\right)\Bigg|_s^\infty dt \\
&= \int_0^\infty \frac{f(t)}{-t}\left(\underbrace{\lim_{s'\to\infty} e^{-s't}}_{=0} - e^{-st}\right)dt \\
&= \int_0^\infty \frac{f(t)}{t}e^{-st}\,dt \\
&= \mathcal{L}\left\{\frac{f(t)}{t}\right\}
\end{aligned}$$

The fifth property is verified. Moving on to property (vi).

Proof of Property VI

$$\begin{aligned}
\mathcal{L}\{tf(t)\} &= \int_0^\infty tf(t)e^{-st}\,dt \\
&= \int_0^\infty f(t)\left(-\frac{d}{ds}\right)e^{-st}\,dt \\
&= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt \\
&= -\frac{d}{ds}\mathcal{L}\{f(t)\} \\
&= -\frac{dF}{ds}
\end{aligned}$$

The sixth property is verified. Moving on to property (vii).

Proof of Property VII

$$\mathcal{L}\{H(t-b)f(t-b)\} = \int_0^\infty H(t-b)f(t-b)e^{-st}\,dt$$

$$\begin{aligned}
\text{Let } u &= t-b \quad \to \quad t = u+b \\
du &= dt
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{H(t-b)f(t-b)\} &= \int_{0-b}^{\infty-b} H(u)f(u)e^{-s(u+b)}\,du \\
&= \int_{-b}^{\infty} H(u)f(u)e^{-su}e^{-sb}\,du \\
&= \int_{-b}^{0} 0 * f(u)e^{-su}e^{-sb}\,du + \int_{0}^{\infty} 1 * f(u)e^{-su}e^{-sb}\,du \\
&= e^{-sb}\int_0^{\infty} f(u)e^{-su}\,du \\
&= e^{-sb}\mathcal{L}\{f(u)\} \\
&= e^{-sb}F(s)
\end{aligned}$$

The seventh property is verified. Moving on to property (viii).

Proof of Property VIII

$$\begin{aligned}
\mathcal{L}\{f(ct)\} &= \int_0^{\infty} f(ct)e^{-st}\,dt \\
\text{Let } u = ct \quad &\rightarrow \quad t = \frac{u}{c} \\
du = cdt \quad &\rightarrow \quad \frac{1}{c}du = dt \\
\mathcal{L}\{f(ct)\} &= \int_{c*0}^{c*\infty} f(u)e^{-s\frac{u}{c}}\,\frac{du}{c} \\
&= \frac{1}{c}\int_0^{\infty} f(u)e^{-\frac{s}{c}u}\,du \\
&= \frac{1}{c}F\left(\frac{s}{c}\right)
\end{aligned}$$

The eighth property is verified. Moving on to property (ix).[1]

Proof of Property IX

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t g(t-t')f(t')\,dt\right\} &= \int_0^{\infty}\int_0^t g(t-t')f(t')\,dt'\,e^{-st}\,dt \\
&= \int_0^{\infty}\int_0^{\infty} g(t-t')f(t')\,dt'\,e^{-st}\,dt
\end{aligned}$$

$$\begin{aligned}
\text{Let } u = t - t' \quad &\rightarrow \quad t = u + t' \\
du = dt &
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t g(t-t')f(t')\,dt\right\} &= \int_0^{\infty}\int_0^{\infty} g(u)f(t')\,dt'\,e^{-s(u+t')}\,du \\
&= \int_0^{\infty}\int_0^{\infty} g(u)f(t')e^{-su}e^{-st'}\,dt'\,du \\
&= \left[\int_0^{\infty} g(u)e^{-su}\,du\right]\left[\int_0^{\infty} f(t')e^{-st'}\,dt'\right] \\
&= G(s)F(s)
\end{aligned}$$

[1] Note that in the context of Laplace transforms, $f(t)$ and $g(t)$ are zero for $t < 0$. Therefore, the limit of integration can be raised from t to ∞.

The ninth and final property is satisfied.