Exercise 1

Verify the entries (2)-(9) in the table of Laplace transforms.

Solution

The table of Laplace transforms the exercise is referring to is given below.

	$f(t)$	$F(s)$
(2)	e^{at}	$\frac{1}{s-a}$
(3)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(4)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(5)	$\cosh at$	$\frac{s}{s^2-a^2}$
(6)	$\sinh at$	$\frac{a}{s^2-a^2}$
(7)	t^k	$\frac{k!}{s^{k+1}}$
(8)	$H(t-b)$	$\frac{1}{s}e^{-bs}$
(9)	$\delta(t-b)$	e^{-bs}

We will verify these transforms using the definition of the Laplace transform one at a time. The Laplace transform of a function, $f(t)$, is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Proof of Transform 2

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}e^{-st}\,dt \\
&= \int_0^\infty e^{-(s-a)t}\,dt \\
&= \frac{1}{-(s-a)}e^{-(s-a)t}\bigg|_0^\infty \\
&= \frac{1}{-(s-a)}\left(\underbrace{\lim_{t\to\infty} e^{-(s-a)t}}_{=\,0} - 1\right) \\
&= \frac{1}{-(s-a)}(-1) \\
&= \frac{1}{s-a}
\end{aligned}$$

The second transform is verified. Moving on to transform 3.

Proof of Transform 3

$$\begin{aligned}
\mathcal{L}\{\cos\omega t\} &= \int_0^\infty \cos\omega t\, e^{-st}\, dt \\
&= \left[\frac{e^{-st}(\omega\sin\omega t - s\cos\omega t)}{s^2+\omega^2}\right]\Bigg|_0^\infty \\
&= \frac{1}{s^2+\omega^2}\left[\underbrace{\lim_{t\to\infty} e^{-st}(\omega\sin\omega t - s\cos\omega t)}_{=\,0} - (-s)\right] \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

The third transform is verified. Moving on to transform 4.

Proof of Transform 4

$$\begin{aligned}
\mathcal{L}\{\sin\omega t\} &= \int_0^\infty \sin\omega t\, e^{-st}\, dt \\
&= \left[-\frac{e^{-st}(\omega\cos\omega t + s\sin\omega t)}{s^2+\omega^2}\right]\Bigg|_0^\infty \\
&= -\frac{1}{s^2+\omega^2}\left[\underbrace{\lim_{t\to\infty} e^{-st}(\omega\cos\omega t + s\sin\omega t)}_{=\,0} - \omega\right] \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

The fourth transform is verified. Moving on to transform 5.

Proof of Transform 5

$$\begin{aligned}
\mathcal{L}\{\cosh at\} &= \int_0^\infty \cosh at\, e^{-st}\, dt \\
&= \left[-\frac{e^{-st}(s\cosh at + a\sinh at)}{s^2-a^2}\right]\Bigg|_0^\infty \\
&= -\frac{1}{s^2-a^2}\left[\underbrace{\lim_{t\to\infty} e^{-st}(s\cosh at + a\sinh at)}_{=\,0} - s\right] \\
&= \frac{s}{s^2-a^2}
\end{aligned}$$

The fifth transform is verified. Moving on to transform 6.

Proof of Transform 6

$$\begin{aligned}
\mathcal{L}\{\sinh at\} &= \int_0^\infty \sinh at\, e^{-st}\, dt \\
&= \left[-\frac{e^{-st}(a\cosh at + s\sinh at)}{s^2 - a^2}\right]\Bigg|_0^\infty \\
&= -\frac{1}{s^2 - a^2}\left[\underbrace{\lim_{t\to\infty} e^{-st}(a\cosh at + s\sinh at)}_{=0} - a\right] \\
&= \frac{a}{s^2 - a^2}
\end{aligned}$$

The sixth transform is verified. Moving on to transform 7.

Proof of Transform 7

$$\mathcal{L}\left\{t^k\right\} = \int_0^\infty t^k e^{-st}\, dt$$

To prove this we will use mathematical induction. The first thing to do is to prove the base case, $k = 0$.

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st}\, dt \\
&= \frac{1}{-s}\left(\underbrace{\lim_{t\to\infty} e^{-st}}_{=0} - 1\right) \\
&= \frac{1}{s} \\
&= \frac{0!}{s^{0+1}}
\end{aligned}$$

Now that the base case has been established, we will assume the inductive hypothesis to be true, namely

$$\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}}.$$

What needs to be shown is that

$$\mathcal{L}\{t^{k+1}\} = \frac{(k+1)!}{s^{k+2}}.$$

$$\begin{aligned}
\mathcal{L}\{t^{k+1}\} &= \int_0^\infty t^{k+1} e^{-st}\, dt \\
&= \int_0^\infty t^k * te^{-st}\, dt \\
&= \int_0^\infty t^k \left(-\frac{d}{ds}\right) e^{-st}\, dt \\
&= -\frac{d}{ds} \int_0^\infty t^k e^{-st}\, dt \\
&= -\frac{d}{ds} \mathcal{L}\{t^k\} \\
&= -\frac{d}{ds} \frac{k!}{s^{k+1}} \\
&= \frac{(k+1)!}{s^{k+2}}
\end{aligned}$$

Therefore, the seventh Laplace transform is true by mathematical induction. Moving on to transform 8.

Proof of Transform 8

$$\begin{aligned}
\mathcal{L}\left\{H(t-b)\right\} &= \int_0^\infty H(t-b) e^{-st}\, dt \\
&= \int_0^b 0 * e^{-st}\, dt + \int_b^\infty 1 * e^{-st}\, dt \\
&= \int_b^\infty e^{-st}\, dt \\
&= \frac{1}{-s} e^{-st}\Big|_b^\infty \\
&= \frac{1}{-s}\left(\underbrace{\lim_{t\to\infty} e^{-st}}_{=0} - e^{-sb}\right) \\
&= \frac{1}{s} e^{-sb}
\end{aligned}$$

The eighth transform is verified. Moving on to transform 9.

Proof of Transform 9

$$\begin{aligned}
\mathcal{L}\left\{\delta(t-b)\right\} &= \int_0^\infty \delta(t-b) e^{-st}\, dt \\
&= e^{-sb}
\end{aligned}$$

This answer follows from the fact that the integral of a function multiplied by a delta function $\delta(t-t_0)$ is simply the function evaluated at t_0, provided that t_0 is within the limits of integration. The ninth and final Laplace transform is verified.