Exercise 4

Show that the Laplace transform of t^k is $\Gamma(k+1)/s^{k+1}$ for any $k > -1$, where $\Gamma(p)$ is the gamma function. (*Hint:* Use property (viii) of the Laplace transform.)

Solution

The Laplace transform of t^k by definition is

$$\mathcal{L}\{t^k\} = \int_0^\infty t^k e^{-st}\,dt.$$

According to Appendix A.5, the Gamma function is defined as

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\,du \quad \text{for } 0 < x < \infty.$$

Thus,

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u}\,du.$$

To get the Laplace transform of t^k to be in terms of the Gamma function, it is necessary to get rid of the t in the exponent of e. We can do this by making a substitution.

$$\begin{aligned} \text{Let } u = st \quad &\rightarrow \quad \frac{1}{s}u = t \\ du = s\,dt \quad &\rightarrow \quad \frac{1}{s}du = dt \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{t^k\} &= \int_{s*0}^{s*\infty} \left(\frac{1}{s}u\right)^k e^{-u}\,\frac{du}{s} \\ &= \int_0^\infty \frac{1}{s^k}u^k e^{-u}\,\frac{du}{s} \\ &= \frac{1}{s^{k+1}}\int_0^\infty u^k e^{-u}\,du \\ &= \frac{1}{s^{k+1}}\Gamma(k+1) \end{aligned}$$

And this is the desired result.