Exercise 5

Use the Laplace transform to solve $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, $u(0, t) = u(l, t) = 0$, $u(x, 0) = \sin(\pi x/l)$, and $u_t(x, 0) = -\sin(\pi x/l)$.

Solution

Let the Laplace transform of a function $u(x, t)$ be defined as

$$\tilde{u}(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^\infty u(x, t)e^{-st} \, dt.$$

Applying the Laplace transform to both sides of the PDE gives

$$\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = \mathcal{L}\left\{c^2 \frac{\partial^2 u}{\partial x^2}\right\}$$

$$s^2 \tilde{u}(x, s) - s u(x, 0) - u_t(x, 0) = c^2 \frac{d^2}{dx^2} \mathcal{L}\{u\}$$

$$s^2 \tilde{u} - s \sin\left(\frac{\pi x}{l}\right) - \left[-\sin\left(\frac{\pi x}{l}\right)\right] = c^2 \frac{d^2 \tilde{u}}{dx^2}$$

$$\frac{d^2 \tilde{u}}{dx^2} - \frac{s^2}{c^2} \tilde{u} = \frac{1}{c^2} (1 - s) \sin \frac{\pi x}{l}$$

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

$$\tilde{u} = \tilde{u}_c + \tilde{u}_p$$

The complementary solution is obtained from solving the associated homogeneous differential equation.

$$\frac{d^2 \tilde{u}_c}{dx^2} - \frac{s^2}{c^2} \tilde{u}_c = 0$$

$$\tilde{u}_c(x, s) = C_1 \cosh \frac{s}{c} x + C_2 \sinh \frac{s}{c} x$$

The constants, $C_1$ and $C_2$, are determined from the given boundary conditions of the problem.

$$\mathcal{L}\{u(0, t)\} = \tilde{u}(0, s) = \mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{u(l, t)\} = \tilde{u}(l, s) = \mathcal{L}\{0\} = 0$$

$$\tilde{u}_c(0, s) = C_1 = 0 \quad \rightarrow \quad C_1 = 0$$

$$\tilde{u}_c(l, s) = C_2 \sinh \frac{s}{c} l = 0 \quad \rightarrow \quad C_2 = 0$$

$$\tilde{u}_c = 0$$

Because the right-hand side of the inhomogeneous differential equation is in terms of $\sin \frac{\pi x}{l}$, we can use the method of undetermined coefficients to find $\tilde{u}_p$. We assume that $\tilde{u}_p = A \cos \frac{\pi x}{l} + B \sin \frac{\pi x}{l}$, and we plug this into the equation to determine the coefficients.

$$-\frac{\pi^2}{l^2} A \cos \frac{\pi x}{l} - \frac{\pi^2}{l^2} B \sin \frac{\pi x}{l} - \frac{s^2}{c^2} A \cos \frac{\pi x}{l} - \frac{s^2}{c^2} B \sin \frac{\pi x}{l} = \frac{1}{c^2} (1 - s) \sin \frac{\pi x}{l}$$

$$\left(-\frac{\pi^2}{l^2} A - \frac{s^2}{c^2} A\right) \cos \pi x + \left(-\frac{\pi^2}{l^2} B - \frac{s^2}{c^2} B\right) \sin \pi x = \frac{1}{c^2} (1 - s) \sin \frac{\pi x}{l}$$

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Matching coefficients on the left and right sides gives

\[-\frac{\pi^2}{l^2} A - \frac{s^2 c^2}{l^2} A = 0 \quad \rightarrow \quad A = 0\]
\[-\frac{\pi^2}{l^2} B - \frac{s^2 c^2}{l^2} B = \frac{1}{c^2} (1 - s) \quad \rightarrow \quad B = \frac{l^2}{s^2 l^2 + c^2 \pi^2} (s - 1).\]

So

\[\tilde{u}_p = \frac{l^2}{s^2 l^2 + c^2 \pi^2} (s - 1) \sin \frac{\pi x}{l}.\]

And the solution to the inhomogeneous differential equation is

\[\tilde{u}(x, s) = \frac{l^2}{s^2 l^2 + c^2 \pi^2} (s - 1) \sin \frac{\pi x}{l}.\]

All that’s left to do now is to take the inverse Laplace transform to find \(u(x, t)\).

\[u(x, t) = \mathcal{L}^{-1} \{ \tilde{u}(x, s) \} = \mathcal{L}^{-1} \left\{ \frac{l^2}{s^2 l^2 + c^2 \pi^2} (s - 1) \sin \frac{\pi x}{l} \right\}\]

\[u(x, t) = \sin \frac{\pi x}{l} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + \frac{c^2 \pi^2}{l^2}} (s - 1) \right\}\]

\[u(x, t) = \sin \frac{\pi x}{l} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \frac{c^2 \pi^2}{l^2}} - \frac{1}{s^2 + \frac{c^2 \pi^2}{l^2}} \right\}\]

\[u(x, t) = \sin \frac{\pi x}{l} \left( \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \frac{c^2 \pi^2}{l^2}} \right\} - \mathcal{L}^{-1} \left\{ \frac{c \pi}{l} \frac{c \pi}{s^2 + \frac{c^2 \pi^2}{l^2}} \right\} \right)\]

\[u(x, t) = \sin \frac{\pi x}{l} \left( \cos \frac{c \pi t}{l} - \frac{1}{c \pi} \sin \frac{c \pi t}{l} \right)\]

Therefore,

\[u(x, t) = \frac{1}{c \pi} \left( c \pi \cos \frac{c \pi t}{l} - l \sin \frac{c \pi t}{l} \right) \sin \frac{\pi x}{l}.\]

We can check to see whether this is the correct solution. Take derivatives of \(u\) with respect to \(x\) and \(t\).

\[u_t = -\frac{1}{l} \left( \frac{\pi}{l} \cos \frac{c \pi t}{l} + c \pi \sin \frac{c \pi t}{l} \right) \sin \frac{\pi x}{l}\]
\[u_{tt} = \frac{c \pi}{l^2} \left( -c \pi \cos \frac{c \pi t}{l} + l \sin \frac{c \pi t}{l} \right) \sin \frac{\pi x}{l}\]
\[u_x = \frac{1}{c l} \left( c \pi \cos \frac{c \pi t}{l} - l \sin \frac{c \pi t}{l} \right) \cos \frac{\pi x}{l}\]
\[u_{xx} = \frac{\pi}{c l^2} \left( -c \pi \cos \frac{c \pi t}{l} + l \sin \frac{c \pi t}{l} \right) \sin \frac{\pi x}{l}\]

\[u_{tt} = c^2 u_{xx}, \text{ so this is indeed the correct solution. By inspection we see that plugging } x = 0 \text{ and } x = l \text{ into } u(x, t) \text{ gives } u = 0. \text{ Also, plugging } t = 0 \text{ into } u(x, t) \text{ and } u_t \text{ gives } u = \sin(\pi x/l) \text{ and } u_t = -\sin(\pi x/l), \text{ respectively, so the initial and boundary conditions are satisfied.}\

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