Exercise 6

Use the Laplace transform to solve

\[ u_{tt} = c^2 u_{xx} + \cos \omega t \sin \pi x \quad \text{for} \quad 0 < x < 1 \]
\[ u(0, t) = u(1, t) = u(x, 0) = u_t(x, 0) = 0. \]

Assume that \( \omega > 0 \) and be careful of the case \( \omega = c\pi \). Check your answer by direct differentiation.

Solution

Let the Laplace transform of a function \( u(x, t) \) be defined as

\[ \hat{u}(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^\infty u(x, t)e^{-st} dt. \]

Applying the Laplace transform to both sides of the PDE gives

\[ \mathcal{L}\left\{ \frac{\partial^2 u}{\partial t^2} \right\} = \mathcal{L}\left\{ c^2 \frac{\partial^2 u}{\partial x^2} + \cos \omega t \sin \pi x \right\} \]
\[ s^2 \hat{u}(x, s) - su(x, 0) - u_t(x, 0) = c^2 \frac{d^2 \hat{u}}{dx^2} + \mathcal{L}\{ \cos \omega t \} \sin \pi x \]
\[ s^2 \hat{u}(x, s) = c^2 \frac{d^2 \hat{u}}{dx^2} + \frac{s}{s^2 + \omega^2} \sin \pi x \]
\[ \frac{d^2 \hat{u}}{dx^2} - \frac{s^2}{c^2} \hat{u} = -\frac{1}{c^2} s \cdot \frac{s}{s^2 + \omega^2} \sin \pi x \]

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

\[ \hat{u} = \hat{u}_c + \hat{u}_p \]

The complementary solution is obtained from solving the associated homogeneous differential equation.

\[ \frac{d^2 \hat{u}_c}{dx^2} - \frac{s^2}{c^2} \hat{u}_c = 0 \]
\[ \hat{u}_c(x, s) = C_1 \cosh \frac{s}{c} x + C_2 \sinh \frac{s}{c} x \]

The constants, \( C_1 \) and \( C_2 \), are determined from the given boundary conditions of the problem.

\[ \mathcal{L}\{u(0, t)\} = \hat{u}(0, s) = \mathcal{L}\{0\} = 0 \]
\[ \mathcal{L}\{u(1, t)\} = \hat{u}(1, s) = \mathcal{L}\{0\} = 0 \]

\[ \hat{u}_c(0, s) = C_1 = 0 \quad \rightarrow \quad C_1 = 0 \]
\[ \hat{u}_c(1, s) = C_2 \sinh \frac{s}{c} = 0 \quad \rightarrow \quad C_2 = 0 \]

\[ \hat{u}_c = 0 \]
Because the right-hand side of the inhomogeneous differential equation is in terms of \( \sin \pi x \), we can use the method of undetermined coefficients to find \( \hat{u}_p \). We assume that 

\[
\hat{u}_p = A \cos \pi x + B \sin \pi x,
\]

and we plug this into the equation to determine the coefficients.

\[
\begin{align*}
-\pi^2 A \cos \pi x - s^2 A \cos \pi x - s^2 B \sin \pi x &= -\frac{1}{c^2} \frac{s}{s^2 + \omega^2} \sin \pi x \\
\left(-\pi^2 A - \frac{s^2}{c^2} A\right) \cos \pi x + \left(-\pi^2 B - \frac{s^2}{c^2} B\right) \sin \pi x &= -\frac{1}{c^2} \frac{s}{s^2 + \omega^2} \sin \pi x
\end{align*}
\]

Matching coefficients on the left and right sides gives

\[
\begin{align*}
-\pi^2 A - \frac{s^2}{c^2} A &= 0 \quad \Rightarrow \quad A = 0 \\
-\pi^2 B - \frac{s^2}{c^2} B &= -\frac{1}{c^2} \frac{s}{s^2 + \omega^2} \quad \Rightarrow \quad B = \frac{1}{s^2 + c^2 \pi^2} \cdot \frac{s}{s^2 + \omega^2}.
\end{align*}
\]

So

\[
\hat{u}_p = \frac{1}{s^2 + c^2 \pi^2} \cdot \frac{s}{s^2 + \omega^2} \sin \pi x.
\]

And the solution to the inhomogeneous differential equation is

\[
\hat{u}(x, s) = \frac{s \sin \pi x}{(s^2 + \omega^2)(s^2 + c^2 \pi^2)}.
\]

All that’s left to do now is to take the inverse Laplace transform to find \( u(x, t) \).

\[
\begin{align*}
u(x, t) &= \mathcal{L}^{-1} \{ \hat{u}(x, s) \} \\
u(x, t) &= \mathcal{L}^{-1} \left\{ \frac{s \sin \pi x}{(s^2 + \omega^2)(s^2 + c^2 \pi^2)} \right\} \\
u(x, t) &= \sin \pi x \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + \omega^2)} \cdot \frac{1}{(s^2 + c^2 \pi^2)} \right\} \\
u(x, t) &= \sin \pi x \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + \omega^2)} \cdot \frac{c\pi}{c\pi (s^2 + c^2 \pi^2)} \right\} \\
u(x, t) &= \frac{\sin \pi x}{c\pi} \int_0^t \sin c\pi (t - t') \cos \omega t' \, dt'
\end{align*}
\]

We have to be careful here since the value of the integral depends on whether \( \omega = c\pi \) or not.

\[
\int_0^t \sin c\pi (t - t') \cos \omega t' \, dt' = \begin{cases} 
\frac{c\pi}{\omega^2 - c^2 \pi^2} (\cos c\pi t - \cos \omega t) & \omega \neq c\pi \\
\frac{1}{2} t \sin c\pi t & \omega = c\pi
\end{cases}
\]

When \( \omega = c\pi \), the phenomenon of resonance occurs, and the amplitude of the wave grows linearly with respect to time. Therefore,

\[
u(x, t) = \begin{cases} 
\frac{\cos c\pi t - \cos \omega t}{\omega^2 - c^2 \pi^2} \sin \pi x & \omega \neq c\pi \\
\frac{t \sin c\pi t}{2c\pi} \sin \pi x & \omega = c\pi
\end{cases}
\]
We can check to see whether this is the correct solution. Take derivatives of $u$ with respect to $x$ and $t$.

\[
\begin{align*}
  u_t &= \begin{cases}
    -c \pi \sin c \pi t + \omega \sin \omega t \quad &\text{if } \omega \neq c \pi \\
    c \pi t \cos \pi t + \sin c \pi t &\text{if } \omega = c \pi
  \end{cases} \sin \pi x
  \\
  u_{tt} &= \begin{cases}
    -c^2 \pi^2 \cos c \pi t + \omega^2 \cos \omega t \quad &\text{if } \omega \neq c \pi \\
    2 \omega \cos c \pi t - c \pi \sin c \pi t &\text{if } \omega = c \pi
  \end{cases} \sin \pi x
  \\
  u_x &= \begin{cases}
    -\pi \cos c \pi t - \cos \omega t \quad &\text{if } \omega \neq c \pi \\
    t \sin c \pi t &\text{if } \omega = c \pi
  \end{cases} \cos \pi x
  \\
  u_{xx} &= \begin{cases}
    -\pi^2 \cos c \pi t - \cos \omega t \quad &\text{if } \omega \neq c \pi \\
    -\pi^2 t \sin c \pi t &\text{if } \omega = c \pi
  \end{cases} \sin \pi x
\end{align*}
\]

And so we have the following.

\[
\begin{align*}
  u_{tt} - c^2 u_{xx} &= \begin{cases}
    -c^2 \pi^2 \cos c \pi t + c^2 \pi^2 \cos c \pi t + (\omega^2 - c^2 \pi^2) \cos \omega t \\
    \omega^2 - c^2 \pi^2
  \end{cases} \sin \pi x = \cos \omega t \sin \pi x \quad &\text{if } \omega \neq c \pi \\
  \frac{2 \cos c \pi t - c \pi t \sin c \pi t + c \pi \sin c \pi t}{2} \sin \pi x = \cos c \pi t \sin \pi x &\text{if } \omega = c \pi
\end{align*}
\]

Thus, $u(x,t)$ satisfies the PDE. By inspection we can see that plugging in $t = 0$, $x = 0$, and $x = 1$ into $u(x,t)$ gives $u = 0$. Also, plugging in $t = 0$ into $u_t$ above gives $u = 0$, so the initial and boundary conditions are satisfied.