Exercise 7

Use the Laplace transform to solve \( u_t = ku_{xx} \) in \((0, l)\), with \( u_x(0, t) = 0, u_x(l, t) = 0\), and \( u(x, 0) = 1 + \cos(2\pi x/l)\).

Solution

Let the Laplace transform of a function \( u(x, t) \) be defined as

\[
\tilde{u}(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^\infty u(x, t)e^{-st}dt.
\]

Applying the Laplace transform to both sides of the PDE gives

\[
\mathcal{L}\left\{ \frac{\partial u}{\partial t} \right\} = \mathcal{L}\left\{ k \frac{\partial^2 u}{\partial x^2} \right\}
\]

\[
s\tilde{u}(x, s) - u(x, 0) = k \frac{d^2}{dx^2} \mathcal{L}\{u\}
\]

\[
s\tilde{u}(x, s) - \left(1 + \cos \frac{2\pi x}{l}\right) = k \frac{d^2}{dx^2} \tilde{u}
\]

\[
\frac{d^2}{dx^2} \tilde{u} - \frac{s}{k} \tilde{u} = -\frac{1}{k} \left(1 + \cos \frac{2\pi x}{l}\right)
\]

What we have is an inhomogeneous ordinary differential equation. The general solution is therefore written as the sum of a complementary solution and a particular solution.

\[
\tilde{u} = \tilde{u}_c + \tilde{u}_p
\]

The complementary solution is obtained from solving the associated homogeneous differential equation.

\[
\frac{d^2}{dx^2} \tilde{u}_c - \frac{s}{k} \tilde{u}_c = 0
\]

\[
\tilde{u}_c(x, s) = C_1 \cosh \sqrt{\frac{s}{k}x} + C_2 \sinh \sqrt{\frac{s}{k}x}
\]

The constants, \(C_1\) and \(C_2\), are determined from the given boundary conditions of the problem.

\[
\mathcal{L}\{u_x(0, t)\} = \tilde{u}_x(0, s) = \mathcal{L}\{0\} = 0
\]

\[
\mathcal{L}\{u_x(l, t)\} = \tilde{u}_x(l, s) = \mathcal{L}\{0\} = 0
\]

\[
\frac{d}{dx} \tilde{u}_c(x, s) = C_1 \sqrt{\frac{s}{k}} \sinh \sqrt{\frac{s}{k}x} + C_2 \sqrt{\frac{s}{k}} \cosh \sqrt{\frac{s}{k}x}
\]

\[
\frac{d}{dx} \tilde{u}_c(0, s) = C_2 \sqrt{\frac{s}{k}} = 0 \quad \rightarrow \quad C_2 = 0
\]

\[
\frac{d}{dx} \tilde{u}_c(l, s) = C_1 \sqrt{\frac{s}{k}} \sinh \sqrt{\frac{s}{k}l} = 0 \quad \rightarrow \quad C_1 = 0
\]

\[
\tilde{u}_c = 0
\]
The variation-of-parameters method can be used to find the particular solution. However, it leads to very complicated integrals. It’s best to split up the particular solution into two parts. That is,

\[ u_p = u_{p1} + u_{p2}, \]

where \( u_{p1} \) and \( u_{p2} \) satisfy

\[
\frac{d^2 u_{p1}}{dx^2} - \frac{s}{k} u_{p1} = -\frac{1}{k} \]

and

\[
\frac{d^2 u_{p2}}{dx^2} - \frac{s}{k} u_{p2} = -\frac{1}{k} \cos \frac{2\pi x}{l} \]

For the first equation, the right-hand side is just a constant, so \( u_{p1} \) is a constant with respect to \( x \). That means the second derivative is 0.

\[
-\frac{s}{k} u_{p1} = -\frac{1}{k} \quad \Rightarrow \quad u_{p1} = \frac{1}{s} \]

For the second equation, the right-hand side is a cosine function, so we use the method of undetermined coefficients to find \( u_{p2} \). We assume that \( u_{p2} = A \cos \frac{2\pi x}{l} + B \sin \frac{2\pi x}{l} \), and we plug this into the equation to determine the coefficients.

\[
-\frac{4\pi^2}{l^2} A \cos \frac{2\pi x}{l} - \frac{4\pi^2}{l^2} B \sin \frac{2\pi x}{l} - \frac{s}{k} A \cos \frac{2\pi x}{l} - \frac{s}{k} B \sin \frac{2\pi x}{l} = -\frac{1}{k} \cos \frac{2\pi x}{l} \]

Matching coefficients on the left and right sides gives

\[
-\frac{4\pi^2}{l^2} A - \frac{s}{k} A = -\frac{1}{k} \quad \Rightarrow \quad A = \frac{l^2}{4\pi^2 k + l^2 s} \]

\[
-\frac{4\pi^2}{l^2} B - \frac{s}{k} B = 0 \quad \Rightarrow \quad B = 0. \]

So

\[ u_{p2} = \frac{l^2}{4\pi^2 k + l^2 s} \cos \frac{2\pi x}{l}. \]

Therefore,

\[ u_p = u_{p1} + u_{p2} = \frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s} \cos \frac{2\pi x}{l}. \]

And the solution to the inhomogeneous differential equation is

\[ u(x, s) = \frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s} \cos \frac{2\pi x}{l}. \]
All that’s left to do now is to take the inverse Laplace transform to find \( u(x,t) \).

\[
\begin{align*}
\mathcal{L}^{-1}\{u(x,s)\} &= u(x,t) \\
\mathcal{L}^{-1}\left\{ \frac{1}{s} + \frac{l^2}{4\pi^2 k + l^2 s} \cos \frac{2\pi x}{l} \right\} \\
\mathcal{L}^{-1}\left\{ \frac{1}{s} + \mathcal{L}^{-1}\left\{ \frac{l^2}{4\pi^2 k + l^2 s} \cos \frac{2\pi x}{l} \right\} \right\} \\
&= 1 + \mathcal{L}^{-1}\left\{ \frac{l^2}{s + \frac{4\pi^2 k}{l^2}} \cos \frac{2\pi x}{l} \right\} \\
&= e^{-\frac{4\pi^2 k}{l^2} t} \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
u(x,t) &= 1 + e^{-\frac{4\pi^2 k}{l^2} t} \cos \frac{2\pi x}{l} \\
\end{align*}
\]

We can check to see whether this is the correct solution. Take derivatives of \( u \) with respect to \( x \) and \( t \).

\[
\begin{align*}
u_t &= -\frac{4\pi^2 k}{l^2} e^{-\frac{4\pi^2 k}{l^2} t} \cos \frac{2\pi x}{l} \\
u_x &= -\frac{2\pi}{l} e^{-\frac{4\pi^2 k}{l^2} t} \sin \frac{2\pi x}{l} \\
u_{xx} &= -\frac{4\pi^2}{l^2} e^{-\frac{4\pi^2 k}{l^2} t} \cos \frac{2\pi x}{l}
\end{align*}
\]

\( u_t = ku_{xx} \), so this is indeed the correct solution. By inspection we see that

\[
\begin{align*}
u(x,0) &= 1 + \cos \frac{2\pi x}{l} \\
\end{align*}
\]

and that plugging in \( x = 0 \) and \( x = l \) into \( u_x \) above gives \( u_x = 0 \). Thus, the initial and boundary conditions are satisfied.