Exercise 1
Derive the continuity equation \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \) from the inhomogeneous Maxwell equations.

Solution

The inhomogeneous equations of Maxwell are

(I) \[ \frac{\partial \mathbf{E}}{\partial t} = c \nabla \times \mathbf{B} - 4\pi \mathbf{J} \]

(II) \[ \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \]

(III) \[ \nabla \cdot \mathbf{E} = 4\pi \rho \]

(IV) \[ \nabla \cdot \mathbf{B} = 0 \].

Differentiate both sides of equation (III) with respect to \( t \).

\[ \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \frac{\partial}{\partial t} (4\pi \rho) \]

\[ \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t} \]

Substitute equation (I) for \( \frac{\partial \mathbf{E}}{\partial t} \).

\[ \nabla \cdot (c \nabla \times \mathbf{B} - 4\pi \mathbf{J}) = 4\pi \frac{\partial \rho}{\partial t} \]

Distribute the divergence operator and bring the constants in front.

\[ c \nabla \cdot (\nabla \times \mathbf{B}) - 4\pi \nabla \cdot \mathbf{J} = 4\pi \frac{\partial \rho}{\partial t} \]

The divergence of a curl is zero, so the first term on the left side vanishes.

\[ -4\pi \nabla \cdot \mathbf{J} = 4\pi \frac{\partial \rho}{\partial t} \]

Divide both sides by \( 4\pi \).

\[ -\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t} \]

Therefore,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \]