Exercise 1

Derive the continuity equation $\partial\rho/\partial t + \nabla \cdot \mathbf{J} = 0$ from the inhomogeneous Maxwell equations.

Solution

The inhomogeneous equations of Maxwell are

$$\text{(I)} \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{J}$$

$$\text{(II)} \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\text{(III)} \quad \nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\text{(IV)} \quad \nabla \cdot \mathbf{B} = 0.$$

Differentiate both sides of equation (III) with respect to t.

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \frac{\partial}{\partial t}(4\pi\rho)$$

$$\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 4\pi\frac{\partial \rho}{\partial t}$$

Substitute equation (I) for $\partial\mathbf{E}/\partial t$.

$$\nabla \cdot (c\nabla \times \mathbf{B} - 4\pi\mathbf{J}) = 4\pi\frac{\partial \rho}{\partial t}$$

Distribute the divergence operator and bring the constants in front.

$$c\nabla \cdot (\nabla \times \mathbf{B}) - 4\pi\nabla \cdot \mathbf{J} = 4\pi\frac{\partial \rho}{\partial t}$$

The divergence of a curl is zero, so the first term on the left side vanishes.

$$-4\pi\nabla \cdot \mathbf{J} = 4\pi\frac{\partial \rho}{\partial t}$$

Divide both sides by 4π.

$$-\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}$$

Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$