Exercise 2

Derive the equations of electrostatics from the Maxwell equations by assuming that $\partial \mathbf{E}/\partial t = \partial \mathbf{B}/\partial t \equiv 0$.

Solution

The inhomogeneous Maxwell equations are

$$\begin{aligned}
\text{(I)} \quad & \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi \mathbf{J} \\
\text{(II)} \quad & \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \\
\text{(III)} \quad & \nabla \cdot \mathbf{E} = 4\pi\rho \\
\text{(IV)} \quad & \nabla \cdot \mathbf{B} = 0.
\end{aligned}$$

Suppose that the electric and magnetic fields are constant with respect to time: $\partial \mathbf{E}/\partial t = \mathbf{0}$ and $\partial \mathbf{B}/\partial t = \mathbf{0}$.

$$\begin{aligned}
\text{(I)} \quad & \mathbf{0} = c\nabla \times \mathbf{B} - 4\pi \mathbf{J} \quad \rightarrow \quad \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \\
\text{(II)} \quad & \mathbf{0} = -c\nabla \times \mathbf{E} \quad \rightarrow \quad \nabla \times \mathbf{E} = \mathbf{0} \\
\text{(III)} \quad & \nabla \cdot \mathbf{E} = 4\pi\rho \\
\text{(IV)} \quad & \nabla \cdot \mathbf{B} = 0
\end{aligned}$$

Electrostatics

The electric field $\mathbf{E}$ is determined uniquely by a curl and a divergence.

$$\begin{aligned}
\nabla \times \mathbf{E} &= \mathbf{0} \\
\nabla \cdot \mathbf{E} &= 4\pi\rho
\end{aligned}$$

This first equation can be satisfied automatically by introducing a potential function $-\phi$ defined by $\mathbf{E} = \nabla(-\phi) = -\nabla\phi$. The minus sign is arbitrary mathematically, but physically it signifies that a charge moving in the direction of an electric field experiences a decrease in potential. Substitute this formula into the second equation.

$$\nabla \cdot (-\nabla\phi) = 4\pi\rho$$

$$-\nabla^2\phi = 4\pi\rho$$

$$\Delta\phi = -4\pi\rho$$

The electric potential therefore satisfies Poisson's equation, where the inhomogeneous term involves ρ, the charge per unit volume (density). Differentiating the divergence of $\mathbf{E}$ with respect to t indicates that ρ is dependent solely on position: $\rho = \rho(x, y, z)$.

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{E}) = \frac{\partial}{\partial t}(4\pi\rho) \quad \rightarrow \quad \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} = 4\pi\frac{\partial \rho}{\partial t} \quad \rightarrow \quad 0 = 4\pi\frac{\partial \rho}{\partial t} \quad \rightarrow \quad \frac{\partial \rho}{\partial t} = 0$$

A Green's function representation for the solution to Poisson's equation can be obtained by using Green's second identity.

$$\iiint\limits_D (u\Delta v - v\Delta u)\, dV = \iint\limits_{\text{bdy } D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

u and v can be any two functions defined on a domain D; let $u = \phi(x, y, z)$ be the electric potential and let $v = G(x, y, z; x_0, y_0, z_0)$ be the Green's function.

$$\iiint\limits_D (\phi\Delta G - G\Delta \phi)\, dV = \iint\limits_{\text{bdy } D} \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right) dS$$

If we require G to satisfy

$$\begin{aligned} \Delta G &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } D \\ G &= 0 \quad \text{at bdy } D, \end{aligned}$$

where (x_0, y_0, z_0) is a point within D, then Green's second identity becomes

$$\iiint\limits_D [\phi(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) - G(x, y, z; x_0, y_0, z_0)(-4\pi\rho)]\, dV = \iint\limits_{\text{bdy } D} \left(\phi\frac{\partial G}{\partial n} - (0)\frac{\partial \phi}{\partial n}\right) dS$$

$$\iiint\limits_D \phi(x, y, z)\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) dV + 4\pi \iiint\limits_D G(x, y, z; x_0, y_0, z_0)\rho(x, y, z)\, dV = \iint\limits_{\text{bdy } D} \phi\frac{\partial G}{\partial n}\, dS.$$

Evaluate the first integral.

$$\phi(x_0, y_0, z_0) + 4\pi \iiint\limits_D G(x, y, z; x_0, y_0, z_0)\rho(x, y, z)\, dV = \iint\limits_{\text{bdy } D} \phi\frac{\partial G}{\partial n}\, dS$$

Take D to be all of space and assume that the potential tends to zero as $|\mathbf{x}| \to \infty$. Then the right side vanishes.

$$\phi(x_0, y_0, z_0) + 4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, z; x_0, y_0, z_0)\rho(x, y, z)\, dx\, dy\, dz = 0$$

Solve for ϕ.

$$\phi(x_0, y_0, z_0) = -4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, z; x_0, y_0, z_0)\rho(x, y, z)\, dx\, dy\, dz$$

Switch the roles of x_0, y_0, and z_0 with those of x, y, and z, respectively.

$$\phi(x, y, z) = -4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_0, y_0, z_0; x, y, z)\rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0$$

Finally, use the fact that the Green's function is symmetric.

$$\phi(x, y, z) = -4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, z; x_0, y_0, z_0)\rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0$$

The aim now is to solve the PDE for G by using a triple Fourier transform; this method is ideal because $-\infty < x, y, z < \infty$. Here we define the triple Fourier transform of a function $F(x, y, z)$ by

$$\mathcal{F}\{F(x,y,z)\} = \overline{F}(k,l,m) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(kx+ly+mz)} F(x,y,z)\, dx\, dy\, dz.$$

As a result, the derivatives of F transform as follows.

$$\mathcal{F}\left\{\frac{\partial^2 F}{\partial x^2}\right\} = (ik)^2 \overline{F}(k,l,m)$$

$$\mathcal{F}\left\{\frac{\partial^2 F}{\partial y^2}\right\} = (il)^2 \overline{F}(k,l,m)$$

$$\mathcal{F}\left\{\frac{\partial^2 F}{\partial z^2}\right\} = (im)^2 \overline{F}(k,l,m)$$

Apply the triple Fourier transform to both sides of the PDE for G.

$$\mathcal{F}\{\Delta G\} = \mathcal{F}\{\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\}$$

Expand the Laplacian operator.

$$\mathcal{F}\left\{\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right\} = \mathcal{F}\{\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\}$$

Use the fact that the triple Fourier transform is a linear operator.

$$\mathcal{F}\left\{\frac{\partial^2 G}{\partial x^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 G}{\partial y^2}\right\} + \mathcal{F}\left\{\frac{\partial^2 G}{\partial z^2}\right\} = \mathcal{F}\{\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\}$$

Transform the derivatives and use the definition on the right side.

$$\begin{aligned}(ik)^2\overline{G}(k,l,m;x_0,y_0,z_0) + (il)^2\overline{G}(k,l,m;x_0,y_0,z_0) + (im)^2\overline{G}(k,l,m;x_0,y_0,z_0)\\ = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(kx+ly+mz)} \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\, dx\, dy\, dz\end{aligned}$$

Simplify both sides.

$$-(k^2+l^2+m^2)\overline{G}(k,l,m;x_0,y_0,z_0) = \frac{1}{(2\pi)^{3/2}} e^{-i(kx_0+ly_0+mz_0)}$$

Solve for $\overline{G}$.

$$\overline{G}(k,l,m;x_0,y_0,z_0) = -\frac{1}{(2\pi)^{3/2}} \frac{e^{-i(kx_0+ly_0+mz_0)}}{k^2+l^2+m^2}$$

Take the inverse triple Fourier transform now to obtain the desired Green's function.

$$\begin{aligned}G(x,y,z;x_0,y_0,z_0) &= \mathcal{F}^{-1}\{\overline{G}(k,l,m;x_0,y_0,z_0)\}\\ &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+ly+mz)}\overline{G}(k,l,m;x_0,y_0,z_0)\, dk\, dl\, dm\\ &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(kx+ly+mz)} \left[-\frac{1}{(2\pi)^{3/2}} \frac{e^{-i(kx_0+ly_0+mz_0)}}{k^2+l^2+m^2}\right] dk\, dl\, dm\\ &= -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i[k(x-x_0)+l(y-y_0)+m(z-z_0)]}}{k^2+l^2+m^2}\, dk\, dl\, dm\end{aligned}$$

Notice that the exponent of e can be written as a dot product of the vectors, $\boldsymbol{\kappa} = \langle k, l, m \rangle$ and $\boldsymbol{\mathfrak{r}} = \langle x - x_0, y - y_0, z - z_0 \rangle$.

$$G(x, y, z; x_0, y_0, z_0) = -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(\boldsymbol{\kappa} \cdot \boldsymbol{\mathfrak{r}})}}{k^2 + l^2 + m^2} \, dk \, dl \, dm$$

Use the definition of the dot product and let α be the angle between the vectors, $\boldsymbol{\kappa}$ and $\boldsymbol{\mathfrak{r}}$.

$$G(x, y, z; x_0, y_0, z_0) = -\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i|\boldsymbol{\kappa}||\boldsymbol{\mathfrak{r}}| \cos\alpha}}{k^2 + l^2 + m^2} \, dk \, dl \, dm$$

Switch to spherical coordinates $(\varrho, \varphi, \theta)$, where θ is the angle from the polar axis.

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) &= -\frac{1}{(2\pi)^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{e^{i\varrho|\boldsymbol{\mathfrak{r}}| \cos\alpha}}{\varrho^2} \, \varrho^2 \sin\theta \, d\varrho \, d\varphi \, d\theta \\ &= -\frac{1}{(2\pi)^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} e^{i\varrho|\boldsymbol{\mathfrak{r}}| \cos\alpha} \sin\theta \, d\varrho \, d\varphi \, d\theta \end{aligned}$$

Orient the polar axis in the direction of $\boldsymbol{\mathfrak{r}}$ so that $\alpha = \theta$. Now the integral can be evaluated with a substitution.

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) &= -\frac{1}{(2\pi)^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} e^{i\varrho|\boldsymbol{\mathfrak{r}}| \cos\theta} \sin\theta \, d\varrho \, d\varphi \, d\theta \\ &= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \left(-\int_0^{\pi} e^{i\varrho|\boldsymbol{\mathfrak{r}}| \cos\theta} \sin\theta \, d\theta \right) d\varphi \, d\varrho \end{aligned}$$

Let $u = \cos\theta$. Then $du = -\sin\theta \, d\theta$.

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) &= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \left(\int_1^{-1} e^{i\varrho|\boldsymbol{\mathfrak{r}}|u} \, du \right) d\varphi \, d\varrho \\ &= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \left(\frac{1}{i\varrho|\boldsymbol{\mathfrak{r}}|} e^{i\varrho|\boldsymbol{\mathfrak{r}}|u} \bigg|_1^{-1} \right) d\varphi \, d\varrho \\ &= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \frac{e^{-i\varrho|\boldsymbol{\mathfrak{r}}|} - e^{i\varrho|\boldsymbol{\mathfrak{r}}|}}{i\varrho|\boldsymbol{\mathfrak{r}}|} \, d\varphi \, d\varrho \\ &= -\frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \frac{2}{\varrho|\boldsymbol{\mathfrak{r}}|} \left(\frac{e^{i\varrho|\boldsymbol{\mathfrak{r}}|} - e^{-i\varrho|\boldsymbol{\mathfrak{r}}|}}{2i} \right) d\varphi \, d\varrho \\ &= -\frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \frac{2 \sin \varrho|\boldsymbol{\mathfrak{r}}|}{\varrho|\boldsymbol{\mathfrak{r}}|} \, d\varphi \, d\varrho \end{aligned}$$

Evaluate the integral in $d\varphi$.

$$\begin{aligned} G(x, y, z; x_0, y_0, z_0) &= -\frac{1}{(2\pi)^3} \int_0^{\infty} (2\pi) \frac{2 \sin \varrho|\boldsymbol{\mathfrak{r}}|}{\varrho|\boldsymbol{\mathfrak{r}}|} \, d\varrho \\ &= -\frac{1}{(2\pi)^2} \int_0^{\infty} \frac{2 \sin \varrho|\boldsymbol{\mathfrak{r}}|}{\varrho|\boldsymbol{\mathfrak{r}}|} \, d\varrho \end{aligned}$$

The integrand is an even function of ϱ, so the factor of 2 can be used to extend the interval of integration to $-\infty < \varrho < \infty$.

$$\begin{aligned}
G(x, y, z; x_0, y_0, z_0) &= -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \frac{\sin \varrho|\boldsymbol{\mathscr{r}}|}{\varrho|\boldsymbol{\mathscr{r}}|}\, d\varrho \\
&= -\frac{1}{(2\pi)^2}\left(\frac{\pi}{|\boldsymbol{\mathscr{r}}|}\right) \\
&= -\frac{1}{4\pi|\boldsymbol{\mathscr{r}}|} \\
&= -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}
\end{aligned}$$

Therefore, the electric potential is

$$\begin{aligned}
\phi(x, y, z) &= -4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y, z; x_0, y_0, z_0)\rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0 \\
&= -4\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[-\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right]\rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\rho(x_0, y_0, z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\, dx_0\, dy_0\, dz_0.
\end{aligned}$$

The electric field can now be obtained by taking the negative gradient of this result.

$$\begin{aligned}
\mathbf{E} &= -\nabla\phi \\
&= -\nabla \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\rho(x_0, y_0, z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\, dx_0\, dy_0\, dz_0 \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nabla\left[\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right]\rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0 \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\langle -\frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}, -\frac{y-y_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}},\right. \\
&\qquad \left. -\frac{z-z_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right\rangle \rho(x_0, y_0, z_0)\, dx_0\, dy_0\, dz_0 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\langle x-x_0, y-y_0, z-z_0\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} \frac{\rho(x_0, y_0, z_0)}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\, dx_0\, dy_0\, dz_0 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\boldsymbol{\mathscr{r}}}\frac{\rho(x_0, y_0, z_0)}{\mathscr{r}^2}\, dV_0 \\
&= \iiint \frac{dq_0}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}
\end{aligned}$$

This is the fabled Coulomb's law.

In conclusion, the explicit formulas for the x-, y-, and z-components of an electrostatic field are

$$E_x(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x-x_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\rho(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$
$$E_y(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y-y_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\rho(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$
$$E_z(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{z-z_0}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\rho(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0.$$

Magnetostatics

The magnetic field is also determined uniquely by a curl and a divergence.

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}$$
$$\nabla\cdot\mathbf{B} = 0$$

The second equation can be satisfied automatically by introducing a potential function $\mathbf{A}$ partly defined by $\mathbf{B} = \nabla\times\mathbf{A}$. Substitute this formula into the first equation.

$$\nabla\times(\nabla\times\mathbf{A}) = \frac{4\pi}{c}\mathbf{J} \tag{1}$$

Analyze the left side.

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{A}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial A_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{A}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mj}\delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3} \boldsymbol{\delta}_m \delta_{mk}\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_i}\frac{\partial A_i}{\partial x_j} - \sum_{i=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_i} \\ &= \sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3}\frac{\partial A_i}{\partial x_i}\right) - \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k A_k\right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}\end{aligned}$$

As a result, equation (1) becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}.$$

To complete the definition of $\mathbf{A}$, we set $\nabla \cdot \mathbf{A} = 0$ so that this equation reduces to

$$-\nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J},$$

or

$$\Delta \mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \quad \Rightarrow \quad \begin{cases} \Delta A_x = -\dfrac{4\pi}{c}J_x \\ \Delta A_y = -\dfrac{4\pi}{c}J_y \\ \Delta A_z = -\dfrac{4\pi}{c}J_z \end{cases}.$$

The magnetic potential, or rather each of its components, therefore satisfies Poisson's equation, where the inhomogeneous term involves $\mathbf{J}$, the current flow per unit area. Differentiating the curl of $\mathbf{B}$ with respect to t indicates that $\mathbf{J}$ is dependent solely on position: $\mathbf{J} = \mathbf{J}(x, y, z)$.

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{\partial}{\partial t}\left(\frac{4\pi}{c}\mathbf{J}\right) \quad \to \quad \nabla \times \frac{\partial \mathbf{B}}{\partial t} = \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t} \quad \to \quad \mathbf{0} = \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t} \quad \to \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$$

A Green's function representation for the solution to Poisson's equation can be obtained by using Green's second identity.

$$\iiint_D (u\Delta v - v\Delta u)\, dV = \iint_{\text{bdy } D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

u and v can be any two functions defined on a domain D; let $u = \mathbf{A}(x, y, z)$ be the magnetic potential and let $v = G(x, y, z; x_0, y_0, z_0)$ be the Green's function.

$$\iiint_D (\mathbf{A}\Delta G - G\Delta \mathbf{A})\, dV = \iint_{\text{bdy } D}\left(\mathbf{A}\frac{\partial G}{\partial n} - G\frac{\partial \mathbf{A}}{\partial n}\right) dS$$

If we require G to satisfy

$$\begin{aligned}\Delta G &= \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{in } D \\ G &= 0 \quad \text{at bdy } D,\end{aligned}$$

where (x_0, y_0, z_0) is a point within D, then Green's second identity becomes

$$\iiint_D \left[\mathbf{A}(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) - G(x,y,z;x_0,y_0,z_0)\left(-\frac{4\pi}{c}\mathbf{J}\right)\right] dV = \iint_{\text{bdy } D} \left(\mathbf{A}\frac{\partial G}{\partial n} - (0)\frac{\partial \mathbf{A}}{\partial n}\right) dS$$

$$\iiint_D \mathbf{A}(x,y,z)\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)dV + \frac{4\pi}{c}\iiint_D G(x,y,z;x_0,y_0,z_0)\mathbf{J}(x,y,z)\,dV = \iint_{\text{bdy } D} \mathbf{A}\frac{\partial G}{\partial n}\,dS.$$

Evaluate the first integral.

$$\mathbf{A}(x_0,y_0,z_0) + \frac{4\pi}{c}\iiint_D G(x,y,z;x_0,y_0,z_0)\mathbf{J}(x,y,z)\,dV = \iint_{\text{bdy } D} \mathbf{A}\frac{\partial G}{\partial n}\,dS$$

Take D to be all of space and assume that each component of the potential tends to zero as $|\mathbf{x}| \to \infty$. Then the right side vanishes.

$$\mathbf{A}(x_0,y_0,z_0) + \frac{4\pi}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z;x_0,y_0,z_0)\mathbf{J}(x,y,z)\,dx\,dy\,dz = \mathbf{0}$$

Solve for $\mathbf{A}$.

$$\mathbf{A}(x_0,y_0,z_0) = -\frac{4\pi}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z;x_0,y_0,z_0)\mathbf{J}(x,y,z)\,dx\,dy\,dz$$

Switch the roles of x_0, y_0, and z_0 with those of x, y, and z, respectively.

$$\mathbf{A}(x,y,z) = -\frac{4\pi}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_0,y_0,z_0;x,y,z)\mathbf{J}(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0$$

Finally, use the fact that the Green's function is symmetric.

$$\begin{aligned}\mathbf{A}(x,y,z) &= -\frac{4\pi}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,z;x_0,y_0,z_0)\mathbf{J}(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 \\ &= -\frac{4\pi}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\right]\mathbf{J}(x_0,y_0,z_0)\,dx_0\,dy_0\,dz_0 \\ &= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathbf{J}(x_0,y_0,z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\,dx_0\,dy_0\,dz_0\end{aligned}$$

The magnetic field can now be obtained by taking the curl of this result.

$$\begin{aligned}\mathbf{B} &= \nabla\times\mathbf{A} \\ &= \nabla\times\frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathbf{J}(x_0,y_0,z_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\,dx_0\,dy_0\,dz_0 \\ &= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nabla\times\frac{\langle J_x(x_0,y_0,z_0), J_y(x_0,y_0,z_0), J_z(x_0,y_0,z_0)\rangle}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\,dx_0\,dy_0\,dz_0\end{aligned}$$

Note that the curl of a function $\mathbf{F}(x,y,z)$ is

$$\nabla\times\mathbf{F} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z\end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}}$$

Evaluating the curl results in

$$\mathbf{B} = \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \frac{J_y(x_0,y_0,z_0)(z-z_0)-J_z(x_0,y_0,z_0)(y-y_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}, \frac{J_z(x_0,y_0,z_0)(x-x_0)-J_x(x_0,y_0,z_0)(z-z_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}, \frac{J_x(x_0,y_0,z_0)(y-y_0)-J_y(x_0,y_0,z_0)(x-x_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\right\rangle dx_0\,dy_0\,dz_0$$

$$= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\langle J_y(x_0,y_0,z_0)(z-z_0)-J_z(x_0,y_0,z_0)(y-y_0), J_z(x_0,y_0,z_0)(x-x_0)-J_x(x_0,y_0,z_0)(z-z_0), J_x(x_0,y_0,z_0)(y-y_0)-J_y(x_0,y_0,z_0)(x-x_0)\rangle \frac{1}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,dx_0\,dy_0\,dz_0$$

$$= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ J_x(x_0,y_0,z_0) & J_y(x_0,y_0,z_0) & J_z(x_0,y_0,z_0)\\ x-x_0 & y-y_0 & z-z_0\end{vmatrix}\frac{1}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}\,dx_0\,dy_0\,dz_0$$

$$= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\mathscr{r}}[\mathbf{J}(x_0,y_0,z_0)\times\boldsymbol{\mathscr{r}}]\frac{1}{\mathscr{r}^2}\,dx_0\,dy_0\,dz_0$$

$$= \frac{1}{c}\iiint \frac{\mathbf{J}(x_0,y_0,z_0)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dV_0.$$

This is the fabled Biot-Savart law. In conclusion, the explicit formulas for the x-, y-, and z-components of a magnetostatic field are

$$B_x(x,y,z) = \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_y(x_0,y_0,z_0)(z-z_0)-J_z(x_0,y_0,z_0)(y-y_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\,dx_0\,dy_0\,dz_0$$

$$B_y(x,y,z) = \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_z(x_0,y_0,z_0)(x-x_0)-J_x(x_0,y_0,z_0)(z-z_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\,dx_0\,dy_0\,dz_0$$

$$B_z(x,y,z) = \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_x(x_0,y_0,z_0)(y-y_0)-J_y(x_0,y_0,z_0)(x-x_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}}\,dx_0\,dy_0\,dz_0.$$