Exercise 3

From $\nabla \cdot \mathbf{B} = 0$ it follows that there exists a vector function $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$. This is a well-known fact in vector analysis; see [EP], [Kr], [Sg1].

(a) Show from Maxwell’s equations that there also exists a scalar function $u$ such that
$$-\nabla u = \mathbf{E} + c^{-1} \partial \mathbf{A} / \partial t.$$

(b) Deduce from (2) that
$$-c^{-1} \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi \rho,$$
and
$$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \nabla \left( \nabla \cdot \mathbf{A} + c^{-1} \frac{\partial u}{\partial t} \right) = \frac{4\pi}{c} \mathbf{J}.$$

(c) Show that if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla \lambda$ and $u$ by $u - (1/c) \partial \lambda / \partial t$, then the equations in parts (a) and (b) are still valid for the new $\mathbf{A}$ and the new $u$. This property is called gauge invariance.

(d) Show that the scalar function $\lambda$ may be chosen so that the new $\mathbf{A}$ and the new $u$ satisfy
$$\nabla \cdot \mathbf{A} + c^{-1} \partial u / \partial t = 0.$$

(e) Conclude that the new potentials satisfy
$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \Delta u = 4\pi \rho \quad \text{and} \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \frac{4\pi}{c} \mathbf{J}.$$

$\mathbf{A}$ is called the vector potential and $u$ the scalar potential. The equations in part (e) are inhomogeneous wave equations. The transformation in part (c) is the simplest example of a gauge transformation.

Solution

Part (a)

Maxwell’s inhomogeneous equations are

(I) $\frac{\partial \mathbf{E}}{\partial t} = c \nabla \times \mathbf{B} - 4\pi \mathbf{J}$

(II) $\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$

(III) $\nabla \cdot \mathbf{E} = 4\pi \rho$

(IV) $\nabla \cdot \mathbf{B} = 0$.

Equation (IV) can be satisfied automatically by introducing a magnetic potential function partly defined by $\nabla \times \mathbf{A} = \mathbf{B}$. The reason this works is because the divergence of any curl is always zero. Substitute this formula into equation (II).

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = -c \nabla \times \mathbf{E}$$

$$\nabla \times \frac{\partial \mathbf{A}}{\partial t} = -c \nabla \times \mathbf{E}$$

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Bring both terms to the left side.

\[ \nabla \times \frac{\partial A}{\partial t} + c \nabla \times E = 0 \]

Divide both sides by \( c \).

\[ \frac{1}{c} \nabla \times \frac{\partial A}{\partial t} + \nabla \times E = 0 \]

\[ \nabla \times \left( \frac{1}{c} \frac{\partial A}{\partial t} + E \right) = 0 \]

This equation can be satisfied automatically by introducing another potential function \(-u\) defined by

\[ \nabla(-u) = \frac{1}{c} \frac{\partial A}{\partial t} + E, \]

or

\[ -\nabla u = E + \frac{1}{c} \frac{\partial A}{\partial t}. \]

The reason this works is because the curl of any gradient of a scalar function is always zero.

**Part (b)**

Take the divergence of both sides of the result from part (a).

\[ \nabla \cdot (-\nabla u) = \nabla \cdot \left( E + \frac{1}{c} \frac{\partial A}{\partial t} \right) \]

\[ -\nabla^2 u = \nabla \cdot E + \frac{1}{c} \nabla \cdot \frac{\partial A}{\partial t} \]

Substitute equation (III) here for the divergence of \( E \).

\[ -\Delta u = 4\pi \rho + \frac{1}{c} \nabla \cdot \frac{\partial A}{\partial t} \]

Therefore, equations (II), (III), and (IV) are encapsulated by

\[ -\frac{1}{c} \nabla \cdot \frac{\partial A}{\partial t} - \Delta u = 4\pi \rho. \]

The aim now is to get the second desired result by starting from equation (I).

\[ \frac{\partial E}{\partial t} = c \nabla \times B - 4\pi J \]

Replace \( B \) with \( \nabla \times A \) and use the result of part (a) to eliminate \( E \).

\[ \frac{\partial}{\partial t} \left( -\nabla u - \frac{1}{c} \frac{\partial A}{\partial t} \right) = c \nabla \times (\nabla \times A) - 4\pi J \]

\[ -\nabla \left( \frac{\partial u}{\partial t} \right) - \frac{1}{c} \frac{\partial^2 A}{\partial t^2} = c \nabla \times (\nabla \times A) - 4\pi J \]

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Divide both sides by $c$.

$$
\frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \nabla \times (\nabla \times A) - \frac{4\pi}{c} J
$$

Solve for $(4\pi/c)J$ and simplify the curl.

$$
\frac{4\pi}{c} J = \frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \nabla \times (\nabla \times A)
$$

$$
= \frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \left( \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \right) \times \left[ \left( \sum_{j=1}^{3} \delta_j \frac{\partial}{\partial x_j} \right) \times \left( \sum_{k=1}^{3} \delta_k \frac{\partial A_k}{\partial x_j} \right) \right]
$$

$$
= \frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \left( \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \right) \times \left[ \left( \sum_{j=1}^{3} \delta_j \frac{\partial}{\partial x_j} \right) \times \left( \sum_{k=1}^{3} \delta_k \frac{\partial A_k}{\partial x_j} \right) \right]
$$

$$
= \frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \left( \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_i \times \delta_j \delta_k \right) \frac{\partial A_k}{\partial x_j}
$$

$$
= \frac{1}{c} \nabla \left( \frac{\partial u}{\partial t} \right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_i \times \delta_j \delta_k \frac{\partial A_k}{\partial x_j}
$$

Therefore, equation (I) becomes

$$
\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial u}{\partial t} \right) = \frac{4\pi}{c} J.
$$
Part (c)

Replace $A$ with $A + \nabla \lambda$ and replace $u$ with $u - (1/c)\partial \lambda / \partial t$ in the result of part (a).

$$-\nabla u = E + \frac{1}{c} \frac{\partial A}{\partial t}$$

$$-\nabla \left( u - \frac{1}{c} \frac{\partial \lambda}{\partial t} \right) = E + \frac{1}{c} \frac{\partial A}{\partial t} (A + \nabla \lambda)$$

$$-\nabla u + \frac{1}{c} \nabla \left( \frac{\partial \lambda}{\partial t} \right) = E + \frac{1}{c} \frac{\partial A}{\partial t} + \frac{1}{c} \nabla \left( \frac{\partial \lambda}{\partial t} \right)$$

Now replace $A$ with $A + \nabla \lambda$ and replace $u$ with $u - (1/c)\partial \lambda / \partial t$ in the first result of part (b).

$$-\frac{1}{c} \nabla \cdot \frac{\partial A}{\partial t} - \Delta u = 4\pi \rho$$

$$-\frac{1}{c} \nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \left( \frac{\partial \lambda}{\partial t} \right) \right) - \Delta \left( u - \frac{1}{c} \frac{\partial \lambda}{\partial t} \right) = 4\pi \rho$$

Now replace $A$ with $A + \nabla \lambda$ and replace $u$ with $u - (1/c)\partial \lambda / \partial t$ in the second result of part (b).

$$\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial u}{\partial t} \right) = \frac{4\pi}{c} J$$

$$\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} + \frac{1}{c^2} \nabla \left( \frac{\partial^2 \lambda}{\partial t^2} \right) = \Delta A + \Delta (\nabla \lambda) + \nabla \left( \nabla \cdot A + \nabla \cdot \nabla \lambda + \frac{1}{c} \frac{\partial u}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} \right) = \frac{4\pi}{c} J$$

$$\frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla \left( \nabla \cdot A + \frac{1}{c} \frac{\partial u}{\partial t} \right) = \frac{4\pi}{c} J$$
Part (d)

In order to complete the definition of $\mathbf{A}$, we require that its divergence satisfies

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t} = 0$$

to simplify the second result of part (b). This is the infamous Lorenz gauge. Let $\mathbf{A}'$ represent the new $\mathbf{A}$, and let $u'$ represent the new $u$.

$$\mathbf{A} = \mathbf{A}' + \nabla \lambda$$
$$u = u' - \frac{1}{c} \frac{\partial \lambda}{\partial t}$$

Take the divergence of both sides of the first equation, and differentiate both sides of the second equation with respect to $t$.

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \nabla \cdot \nabla \lambda$$
$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t} - \frac{1}{c} \frac{\partial^2 \lambda}{\partial t^2}$$

Divide both sides of the second equation by $c$.

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \Delta \lambda$$
$$\frac{1}{c} \frac{\partial u}{\partial t} = \frac{1}{c} \frac{\partial u'}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2}$$

Add the respective sides of each equation.

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t} = \left( \nabla \cdot \mathbf{A}' + \frac{1}{c} \frac{\partial u'}{\partial t} \right) + \Delta \lambda - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2}$$

Solve for the primed variables in parentheses.

$$\nabla \cdot \mathbf{A}' + \frac{1}{c} \frac{\partial u'}{\partial t} = -\Delta \lambda + \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t}$$

Choose $\lambda$ so that the right side is zero.

$$-\Delta \lambda + \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t} = 0$$

For the Lorenz gauge to hold then, $\lambda$ must be related to the old variables by

$$\Delta \lambda - \frac{1}{c^2} \frac{\partial^2 \lambda}{\partial t^2} = \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t}.$$
Part (e)

With the Lorenz gauge, the second result of part (b) reduces to

\[
\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \nabla \left( \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial u}{\partial t} \right) = \frac{4\pi}{c} \mathbf{J} \quad \Rightarrow \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \frac{4\pi}{c} \mathbf{J} \quad \Rightarrow \quad \begin{cases}
\frac{1}{c^2} \frac{\partial^2 A_x}{\partial t^2} - \Delta A_x = \frac{4\pi}{c} J_x \\
\frac{1}{c^2} \frac{\partial^2 A_y}{\partial t^2} - \Delta A_y = \frac{4\pi}{c} J_y \\
\frac{1}{c^2} \frac{\partial^2 A_z}{\partial t^2} - \Delta A_z = \frac{4\pi}{c} J_z
\end{cases}
\]

and the first result of part (b) reduces to

\[
-\frac{1}{c} \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi \rho \quad \Rightarrow \quad -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) - \Delta u = 4\pi \rho \quad \Rightarrow \quad -\frac{1}{c} \frac{\partial}{\partial t} \left( -\frac{1}{c} \frac{\partial u}{\partial t} \right) - \Delta u = 4\pi \rho
\]

Therefore, by introducing the potential functions, \( u \) and \( \mathbf{A} \), Maxwell’s equations reduce to a system of four decoupled three-dimensional inhomogeneous wave equations. Once they’re solved for, the electric and magnetic fields are obtained by

\[
\mathbf{E} = -\nabla u - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} = \nabla \times \mathbf{A}.
\]