Exercise 5

Derive carefully the formulas (8) and (9) for the solution of Maxwell’s equations.

Solution

From the homogeneous Maxwell equations,

(I) \( \frac{\partial E}{\partial t} = c\nabla \times B \)

(II) \( \frac{\partial B}{\partial t} = -c\nabla \times E \)

(III) \( \nabla \cdot E = 0 \)

(IV) \( \nabla \cdot B = 0 \),

it follows that the electric and magnetic fields satisfy the homogeneous three-dimensional wave equation.

\[
\frac{\partial^2 E}{\partial t^2} = c^2 \Delta E, \quad -\infty < x, y, z < \infty, \ t > 0
\]

\[
E(x, y, z, 0) = E^0(x, y, z)
\]

\[
\frac{\partial E}{\partial t}(x, y, z, 0) = c\nabla \times B^0(x, y, z)
\]

\[
\frac{\partial^2 B}{\partial t^2} = c^2 \Delta B, \quad -\infty < x, y, z < \infty, \ t > 0
\]

\[
B(x, y, z, 0) = B^0(x, y, z)
\]

\[
\frac{\partial B}{\partial t}(x, y, z, 0) = -c\nabla \times E^0(x, y, z)
\]
Their solutions are given by the formula of Kirchhoff and Poisson.

\[
E(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} E^0(x_0, y_0, z_0) \, dS_0 \right] \quad + \quad \frac{1}{4\pi c^2 t} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} [c\nabla \times B^0(x_0, y_0, z_0)] \, dS_0
\]

\[
B(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} B^0(x_0, y_0, z_0) \, dS_0 \right] \quad + \quad \frac{1}{4\pi c^2 t} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} [-c\nabla \times E^0(x_0, y_0, z_0)] \, dS_0
\]

Write the surface integrals in square brackets explicitly by using spherical coordinates \((r_0, \phi_0, \theta_0)\), where \(\theta_0\) is the angle from the polar axis.

\[
x_0 - x = ct \sin \theta_0 \cos \phi_0 \\
y_0 - y = ct \sin \theta_0 \sin \phi_0 \\
z_0 - z = ct \cos \phi_0
\]

As a result,

\[
E(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} E^0(ct, \phi_0, \theta_0)(c^2 t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) \right] \quad + \quad \frac{1}{4\pi ct} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} \nabla \times B^0(x_0, y_0, z_0) \, dS_0
\]

\[
B(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} B^0(ct, \phi_0, \theta_0)(c^2 t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) \right] \quad - \quad \frac{1}{4\pi ct} \int\int_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2 t^2} \nabla \times E^0(x_0, y_0, z_0) \, dS_0.
\]
Bring the constants out in front.

\[
\mathbf{E}(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{t}{4\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{E}^0(\alpha, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] + \frac{1}{4\pi c t} \int \int \nabla \times \mathbf{B}^0(x_0, y_0, z_0) \, dS_0 \\
\mathbf{B}(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{t}{4\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{B}^0(\alpha, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] - \frac{1}{4\pi c t} \int \int \nabla \times \mathbf{E}^0(x_0, y_0, z_0) \, dS_0
\]

Evaluate the derivative by using the product and chain rules.

\[
\mathbf{E}(x, y, z, t) = \left[ \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{E}^0(\alpha, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 + \frac{t}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\partial \mathbf{E}^0}{\partial \alpha}(\alpha, \phi_0, \theta_0) \frac{\partial (\alpha)}{\partial t} (\alpha) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] \\
+ \frac{1}{4\pi c t} \int \int \nabla \times \mathbf{B}^0(x_0, y_0, z_0) \, dS_0 \\
\mathbf{B}(x, y, z, t) = \left[ \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \mathbf{B}^0(\alpha, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 + \frac{t}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\partial \mathbf{B}^0}{\partial \alpha}(\alpha, \phi_0, \theta_0) \frac{\partial (\alpha)}{\partial t} (\alpha) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] \\
- \frac{1}{4\pi c t} \int \int \nabla \times \mathbf{E}^0(x_0, y_0, z_0) \, dS_0
\]
Evaluate the remaining derivative.

\[
E(x, y, z, t) = \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} E^0(\ct, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial E^0(\ct, \phi_0, \theta_0)}{\partial r_0} (c) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] \\
+ \frac{1}{4\pi \ct} \iiint_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2} \nabla \times B^0(x_0, y_0, z_0) \, dS_0
\]

\[
B(x, y, z, t) = \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} B^0(\ct, \phi_0, \theta_0) \sin \theta_0 \, d\phi_0 \, d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial B^0(\ct, \phi_0, \theta_0)}{\partial r_0} (c) \sin \theta_0 \, d\phi_0 \, d\theta_0 \right] \\
- \frac{1}{4\pi \ct} \iiint_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2} \nabla \times E^0(x_0, y_0, z_0) \, dS_0
\]

Factor \(1/(4\pi ct)\) from the square brackets and prepare to write the differential as \(dS_0\) again.

\[
E(x, y, z, t) = \frac{1}{4\pi \ct} \left[ \frac{1}{ct} \int_0^\pi \int_0^{2\pi} E^0(\ct, \phi_0, \theta_0) (c^2/t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) + \int_0^\pi \int_0^{2\pi} \frac{\partial E^0(\ct, \phi_0, \theta_0)}{\partial r_0} (c^2/t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) \right] \\
+ \frac{1}{4\pi \ct} \iiint_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2} \nabla \times B^0(x_0, y_0, z_0) \, dS_0
\]

\[
B(x, y, z, t) = \frac{1}{4\pi \ct} \left[ \frac{1}{ct} \int_0^\pi \int_0^{2\pi} B^0(\ct, \phi_0, \theta_0) (c^2/t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) + \int_0^\pi \int_0^{2\pi} \frac{\partial B^0(\ct, \phi_0, \theta_0)}{\partial r_0} (c^2/t^2 \sin \theta_0 \, d\phi_0 \, d\theta_0) \right] \\
- \frac{1}{4\pi \ct} \iiint_{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2} \nabla \times E^0(x_0, y_0, z_0) \, dS_0
\]
Change the integrals in square brackets back to Cartesian coordinates.

\[
E(x,y,z,t) = \frac{1}{4\pi ct} \left[ \frac{1}{ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi E^0(x_0,y_0,z_0) dS_0 + \frac{\partial E^0}{\partial r_0}(x_0,y_0,z_0) dS_0 \right] + \frac{1}{4\pi ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi \nabla \times B^0(x_0,y_0,z_0) dS_0
\]

\[
B(x,y,z,t) = \frac{1}{4\pi ct} \left[ \frac{1}{ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi B^0(x_0,y_0,z_0) dS_0 + \frac{\partial B^0}{\partial r_0}(x_0,y_0,z_0) dS_0 \right] - \frac{1}{4\pi ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi \nabla \times E^0(x_0,y_0,z_0) dS_0
\]

Therefore, combining the integrals, equations (8) and (9) for the electric and magnetic fields are obtained.

\[
E(x,y,z,t) = \frac{1}{4\pi ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi \left[ \frac{1}{ct} E^0(x_0,y_0,z_0) + \frac{\partial E^0}{\partial r_0}(x_0,y_0,z_0) + \nabla \times B^0(x_0,y_0,z_0) \right] dS_0
\]

\[
B(x,y,z,t) = \frac{1}{4\pi ct} \int_0^\infty \int_0^{2\pi} \int_0^\pi \left[ \frac{1}{ct} B^0(x_0,y_0,z_0) + \frac{\partial B^0}{\partial r_0}(x_0,y_0,z_0) - \nabla \times E^0(x_0,y_0,z_0) \right] dS_0
\]