Exercise 5

Derive carefully the formulas (8) and (9) for the solution of Maxwell's equations.

Solution

From the homogeneous Maxwell equations,

$$\begin{aligned}
&\text{(I)} \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} \\
&\text{(II)} \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \\
&\text{(III)} \quad \nabla \cdot \mathbf{E} = 0 \\
&\text{(IV)} \quad \nabla \cdot \mathbf{B} = 0,
\end{aligned}$$

it follows that the electric and magnetic fields satisfy the homogeneous three-dimensional wave equation.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) &= \frac{\partial}{\partial t}(c\nabla \times \mathbf{B}) & \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) &= \frac{\partial}{\partial t}(-c\nabla \times \mathbf{E}) \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c\nabla \times \frac{\partial \mathbf{B}}{\partial t} & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c\nabla \times \frac{\partial \mathbf{E}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c\nabla \times (-c\nabla \times \mathbf{E}) & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c\nabla \times (c\nabla \times \mathbf{B}) \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2\nabla \times (\nabla \times \mathbf{E}) & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2\nabla \times (\nabla \times \mathbf{B}) \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2[\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}] & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B}] \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2[\nabla(0) - \Delta\mathbf{E}] & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2[\nabla(0) - \Delta\mathbf{B}] \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2\Delta\mathbf{E} & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= c^2\Delta\mathbf{B}
\end{aligned}$$

The aim here is to solve them in all of space, each with two prescribed initial conditions.

$$\begin{aligned}
&\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2\Delta\mathbf{E}, \quad -\infty < x, y, z < \infty,\ t > 0 & &\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2\Delta\mathbf{B}, \quad -\infty < x, y, z < \infty,\ t > 0 \\
&\mathbf{E}(x, y, z, 0) = \mathbf{E}^0(x, y, z) & &\mathbf{B}(x, y, z, 0) = \mathbf{B}^0(x, y, z) \\
&\frac{\partial \mathbf{E}}{\partial t}(x, y, z, 0) = c\nabla \times \mathbf{B}^0(x, y, z) & &\frac{\partial \mathbf{B}}{\partial t}(x, y, z, 0) = -c\nabla \times \mathbf{E}^0(x, y, z)
\end{aligned}$$

Their solutions are given by the formula of Kirchhoff and Poisson.

$$\mathbf{E}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{E}^0(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} [c\nabla\times\mathbf{B}^0(x_0,y_0,z_0)]\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{B}^0(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} [-c\nabla\times\mathbf{E}^0(x_0,y_0,z_0)]\,dS_0$$

Write the surface integrals in square brackets explicitly by using spherical coordinates (r_0,ϕ_0,θ_0), where θ_0 is the angle from the polar axis.

$$\begin{aligned} x_0 - x &= ct\sin\theta_0\cos\phi_0 \\ y_0 - y &= ct\sin\theta_0\sin\phi_0 \\ z_0 - z &= ct\cos\phi_0 \end{aligned}$$

As a result,

$$\mathbf{E}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_0^\pi\int_0^{2\pi} \mathbf{E}^0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_0^\pi\int_0^{2\pi} \mathbf{B}^0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0.$$

Bring the constants out in front.

$$\mathbf{E}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right] + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right] - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0$$

Evaluate the derivative by using the product and chain rules.

$$\begin{aligned}\mathbf{E}(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\sin\theta_0\,d\phi_0\,d\theta_0\right]\\ + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0\end{aligned}$$

$$\begin{aligned}\mathbf{B}(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial\mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\sin\theta_0\,d\phi_0\,d\theta_0\right]\\ - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0\end{aligned}$$

Evaluate the remaining derivative.

$$\mathbf{E}(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\mathbf{E}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{\partial\mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0)(c)\sin\theta_0\,d\phi_0\,d\theta_0\right] + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \left[\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\mathbf{B}^0(ct,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{t}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{\partial\mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0)(c)\sin\theta_0\,d\phi_0\,d\theta_0\right] - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0$$

Factor $1/(4\pi ct)$ from the square brackets and prepare to write the differential as dS_0 again.

$$\mathbf{E}(x,y,z,t) = \frac{1}{4\pi ct}\left[\frac{1}{ct}\int_0^\pi\int_0^{2\pi}\mathbf{E}^0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0) + \int_0^\pi\int_0^{2\pi}\frac{\partial\mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{1}{4\pi ct}\left[\frac{1}{ct}\int_0^\pi\int_0^{2\pi}\mathbf{B}^0(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0) + \int_0^\pi\int_0^{2\pi}\frac{\partial\mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0)(c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)\right] - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}}\nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0$$

Change the integrals in square brackets back to Cartesian coordinates.

$$\mathbf{E}(x,y,z,t) = \frac{1}{4\pi ct}\left[\frac{1}{ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{E}^0(x_0,y_0,z_0)\,dS_0 + \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{E}^0}{\partial r_0}(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \nabla\times\mathbf{B}^0(x_0,y_0,z_0)\,dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{1}{4\pi ct}\left[\frac{1}{ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \mathbf{B}^0(x_0,y_0,z_0)\,dS_0 + \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{B}^0}{\partial r_0}(x_0,y_0,z_0)\,dS_0\right] - \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \nabla\times\mathbf{E}^0(x_0,y_0,z_0)\,dS_0$$

Therefore, combining the integrals, equations (8) and (9) for the electric and magnetic fields are obtained.

$$\mathbf{E}(x,y,z,t) = \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct}\mathbf{E}^0(x_0,y_0,z_0) + \frac{\partial \mathbf{E}^0}{\partial r_0}(x_0,y_0,z_0) + \nabla\times\mathbf{B}^0(x_0,y_0,z_0)\right] dS_0$$

$$\mathbf{B}(x,y,z,t) = \frac{1}{4\pi ct}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct}\mathbf{B}^0(x_0,y_0,z_0) + \frac{\partial \mathbf{B}^0}{\partial r_0}(x_0,y_0,z_0) - \nabla\times\mathbf{E}^0(x_0,y_0,z_0)\right] dS_0$$

www.stemjock.com