Exercise 7

Prove that (3) follows directly from (8)-(9).

Solution

From the homogeneous Maxwell equations,

$$\text{(I)} \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B}$$

$$\text{(II)} \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\text{(III)} \quad \nabla \cdot \mathbf{E} = 0$$

$$\text{(IV)} \quad \nabla \cdot \mathbf{B} = 0,$$

it follows that the electric and magnetic fields satisfy the homogeneous three-dimensional wave equation. Suppose that they are each subject to two known initial conditions.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \Delta \mathbf{E}, \quad -\infty < x, y, z < \infty,\ t > 0 \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \Delta \mathbf{B}, \quad -\infty < x, y, z < \infty,\ t > 0$$

$$\mathbf{E}(x, y, z, 0) = \mathbf{E}^0(x, y, z) \qquad \mathbf{B}(x, y, z, 0) = \mathbf{B}^0(x, y, z)$$

$$\frac{\partial \mathbf{E}}{\partial t}(x, y, z, 0) = c\nabla \times \mathbf{B}^0(x, y, z) \qquad \frac{\partial \mathbf{B}}{\partial t}(x, y, z, 0) = -c\nabla \times \mathbf{E}^0(x, y, z)$$

Their solution is given by equations (8) and (9) in the textbook.

$$\mathbf{E}(x, y, z, t) = \frac{1}{4\pi ct} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct}\mathbf{E}^0(x_0, y_0, z_0) + \frac{\partial \mathbf{E}^0}{\partial r_0}(x_0, y_0, z_0) + \nabla \times \mathbf{B}^0(x_0, y_0, z_0)\right] dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{1}{4\pi ct} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct}\mathbf{B}^0(x_0, y_0, z_0) + \frac{\partial \mathbf{B}^0}{\partial r_0}(x_0, y_0, z_0) - \nabla \times \mathbf{E}^0(x_0, y_0, z_0)\right] dS_0$$

Distribute $1/(4\pi ct)$ and write out the integrals explicitly by using spherical coordinates (r_0, ϕ_0, θ_0), where θ_0 is the angle from the polar axis.

$$x_0 - x = ct \sin\theta_0 \cos\phi_0$$
$$y_0 - y = ct \sin\theta_0 \sin\phi_0$$
$$z_0 - z = ct \cos\phi_0$$

As a result,

$$\mathbf{E}(x,y,z,t) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi c^2t^2}\mathbf{E}^0(ct,\phi_0,\theta_0) + \frac{1}{4\pi ct}\frac{\partial \mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0) + \frac{1}{4\pi ct}\nabla\times\mathbf{B}^0(ct,\phi_0,\theta_0)\right](c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0)$$

$$\mathbf{B}(x,y,z,t) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi c^2t^2}\mathbf{B}^0(ct,\phi_0,\theta_0) + \frac{1}{4\pi ct}\frac{\partial \mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0) - \frac{1}{4\pi ct}\nabla\times\mathbf{E}^0(ct,\phi_0,\theta_0)\right](c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0),$$

or after simplifying,

$$\mathbf{E}(x,y,z,t) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{E}^0(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\nabla\times\mathbf{B}^0(ct,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0 \tag{5}$$

$$\mathbf{B}(x,y,z,t) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{B}^0(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0) - \frac{ct}{4\pi}\nabla\times\mathbf{E}^0(ct,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0. \tag{6}$$

Taking the limit of both sides as $t \to 0$, the latter two terms in each integrand vanish.

$$\mathbf{E}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{E}^0(0,\phi_0,\theta_0) + 0 + 0\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\mathbf{B}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{B}^0(0,\phi_0,\theta_0) + 0 - 0\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

Since the integrals are over the sphere defined by $(x_0-x)^2+(y_0-y)^2+(z_0-z)^2 = c^2t^2$, $x_0 = x$ and $y_0 = y$ and $z_0 = z$ if $t = 0$.

$$\mathbf{E}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{E}^0(x,y,z)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\mathbf{B}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{B}^0(x,y,z)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

Bring the constants in front and evaluate the integrals.

$$\mathbf{E}(x,y,z,0) = \frac{1}{4\pi}\mathbf{E}^0(x,y,z)\left(\int_0^{\pi}\sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi}d\phi_0\right) = \frac{1}{4\pi}\mathbf{E}^0(x,y,z)(2)(2\pi)$$

$$\mathbf{B}(x,y,z,0) = \frac{1}{4\pi}\mathbf{B}^0(x,y,z)\left(\int_0^{\pi}\sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi}d\phi_0\right) = \frac{1}{4\pi}\mathbf{B}^0(x,y,z)(2)(2\pi)$$

Therefore, the initial conditions in equation (3) of the textbook are obtained.

$$\mathbf{E}(x,y,z,0) = \mathbf{E}^0(x,y,z)$$

$$\mathbf{B}(x,y,z,0) = \mathbf{B}^0(x,y,z)$$

Differentiate both sides of equation (5) and equation (6) with respect to t in order to check the second initial condition for $\mathbf{E}$ and $\mathbf{B}$, respectively.

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{E}^0(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\nabla\times\mathbf{B}^0(ct,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$
$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{4\pi}\mathbf{B}^0(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0) - \frac{ct}{4\pi}\nabla\times\mathbf{E}^0(ct,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

Bring the time derivative inside and use the product and chain rules.

$$\begin{aligned}\frac{\partial \mathbf{E}}{\partial t} = \int_0^{\pi}\int_0^{2\pi}\bigg[&\frac{1}{4\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct) + \frac{c}{4\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial^2 \mathbf{E}^0}{\partial r_0^2}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\\ &+ \frac{c}{4\pi}\nabla\times\mathbf{B}^0(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial}{\partial r_0}\nabla\times\mathbf{B}^0(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\bigg]\sin\theta_0\,d\phi_0\,d\theta_0\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} = \int_0^{\pi}\int_0^{2\pi}\bigg[&\frac{1}{4\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct) + \frac{c}{4\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(ct,\phi_0,\theta_0) + \frac{ct}{4\pi}\frac{\partial^2 \mathbf{B}^0}{\partial r_0^2}(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\\ &- \frac{c}{4\pi}\nabla\times\mathbf{E}^0(ct,\phi_0,\theta_0) - \frac{ct}{4\pi}\frac{\partial}{\partial r_0}\nabla\times\mathbf{E}^0(ct,\phi_0,\theta_0)\frac{\partial}{\partial t}(ct)\bigg]\sin\theta_0\,d\phi_0\,d\theta_0\end{aligned}$$

Set $t=0$ and simplify the integrands.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{2c}{4\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(0,\phi_0,\theta_0) + \frac{c}{4\pi}\nabla\times\mathbf{B}^0(0,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \int_0^{\pi}\int_0^{2\pi}\left[\frac{2c}{4\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(0,\phi_0,\theta_0) - \frac{c}{4\pi}\nabla\times\mathbf{E}^0(0,\phi_0,\theta_0)\right]\sin\theta_0\,d\phi_0\,d\theta_0$$

Split each of the integrals into two.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial \mathbf{E}^0}{\partial r_0}(0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 + \frac{c}{4\pi}\int_0^{\pi}\int_0^{2\pi}\nabla\times\mathbf{B}^0(0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\int_0^{\pi}\int_0^{2\pi}\frac{\partial \mathbf{B}^0}{\partial r_0}(0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0 - \frac{c}{4\pi}\int_0^{\pi}\int_0^{2\pi}\nabla\times\mathbf{E}^0(0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0$$

Pull the derivative with respect to r_0 out in front in the first ones. In the second ones, note that since the integrals are over the sphere defined by $(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2$, $x_0=x$ and $y_0=y$ and $z_0=z$ if $t=0$.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\int_0^{\pi}\int_0^{2\pi}\mathbf{E}^0(r_0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right]\bigg|_{r_0=0} + \frac{c}{4\pi}\int_0^{\pi}\int_0^{2\pi}\nabla\times\mathbf{B}^0(x,y,z)\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\int_0^{\pi}\int_0^{2\pi}\mathbf{B}^0(r_0,\phi_0,\theta_0)\sin\theta_0\,d\phi_0\,d\theta_0\right]\bigg|_{r_0=0} - \frac{c}{4\pi}\int_0^{\pi}\int_0^{2\pi}\nabla\times\mathbf{E}^0(x,y,z)\sin\theta_0\,d\phi_0\,d\theta_0$$

Write the first integrals as surface integrals and pull the constants in front of the second integrals.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\int_0^{\pi}\int_0^{2\pi}\mathbf{E}^0(r_0,\phi_0,\theta_0)(r_0^2\sin\theta_0\,d\phi_0\,d\theta_0)\right]\bigg|_{r_0=0} + \frac{c\nabla\times\mathbf{B}^0(x,y,z)}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sin\theta_0\,d\phi_0\,d\theta_0$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\int_0^{\pi}\int_0^{2\pi}\mathbf{B}^0(r_0,\phi_0,\theta_0)(r_0^2\sin\theta_0\,d\phi_0\,d\theta_0)\right]\bigg|_{r_0=0} - \frac{c\nabla\times\mathbf{E}^0(x,y,z)}{4\pi}\int_0^{\pi}\int_0^{2\pi}\sin\theta_0\,d\phi_0\,d\theta_0$$

Change the first integrals back to Cartesian coordinates and write the second ones as iterated integrals.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=r_0^2}}\mathbf{E}^0(x_0,y_0,z_0)\,dS_0\right]\bigg|_{r_0=0} + \frac{c\nabla\times\mathbf{B}^0(x,y,z)}{4\pi}\left(\int_0^{\pi}\sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi}d\phi_0\right)$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=r_0^2}}\mathbf{B}^0(x_0,y_0,z_0)\,dS_0\right]\bigg|_{r_0=0} - \frac{c\nabla\times\mathbf{E}^0(x,y,z)}{4\pi}\left(\int_0^{\pi}\sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi}d\phi_0\right)$$

Use the divergence theorem in the first integrals and evaluate the second integrals.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le r_0^2}}\nabla\cdot\mathbf{E}^0(x_0,y_0,z_0)\,dV_0\right]\bigg|_{r_0=0} + \frac{c\nabla\times\mathbf{B}^0(x,y,z)}{4\pi}(2)(2\pi)$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = \frac{c}{2\pi}\frac{\partial}{\partial r_0}\left[\frac{1}{r_0^2}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le r_0^2}}\nabla\cdot\mathbf{B}^0(x_0,y_0,z_0)\,dV_0\right]\bigg|_{r_0=0} - \frac{c\nabla\times\mathbf{E}^0(x,y,z)}{4\pi}(2)(2\pi)$$

Since the provided electric and magnetic fields at $t=0$ satisfy equations (III) and (IV), these first integrals are zero. Therefore,

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = c\nabla\times\mathbf{B}^0(x,y,z)$$

$$\frac{\partial \mathbf{B}}{\partial t}(x,y,z,0) = -c\nabla\times\mathbf{E}^0(x,y,z).$$