Exercise 3

Solve the nonlinear equation $u_t + uu_x = 0$ with the auxiliary condition $u(x, 0) = x$. Sketch some of the characteristic lines.

Solution

For a function of two variables $u = u(x, t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Comparing the right side with the given PDE, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = u, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 0, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. Equation (2) tells us that u is independent of t, so the characteristic curves can be obtained by integrating both sides of equation (1) with respect to t.

$$x = ut + \xi$$

Solve this equation for ξ.

$$\xi = x - ut$$

The general solution to the PDE is then

$$u(x, t) = f(x - ut).$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x, 0) = f(x) = x$$

Though this is in terms of x, the equation is actually $f(w) = w$, where w is any expression we choose: $f(x - ut) = x - ut$. Thus,

$$u(x, t) = x - ut.$$

Solve this equation for u.

$$\begin{aligned} u &= x - ut \\ u + ut &= x \\ (1 + t)u &= x \end{aligned}$$

Therefore,

$$\boxed{u(x,t) = \frac{x}{1+t}.}$$

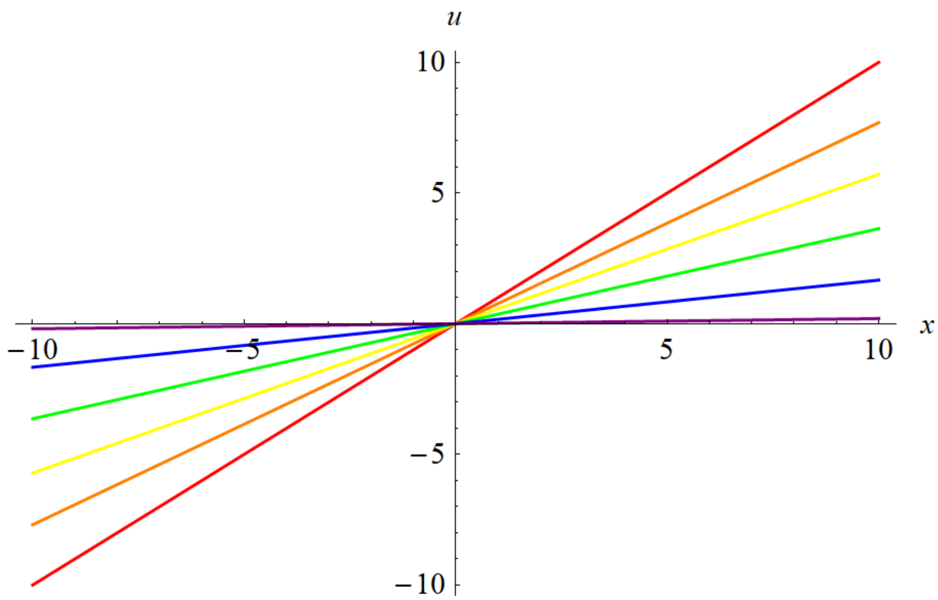

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.3$, $t = 0.75$, $t = 1.75$, $t = 5$, $t = 50$, respectively.

Recall from section 1.3 that the simple transport equation is $u_t + cu_x = 0$. Comparing the PDE we have to this, we see that the speed c is equal to $2u$. The higher u is, the faster to the right it travels.

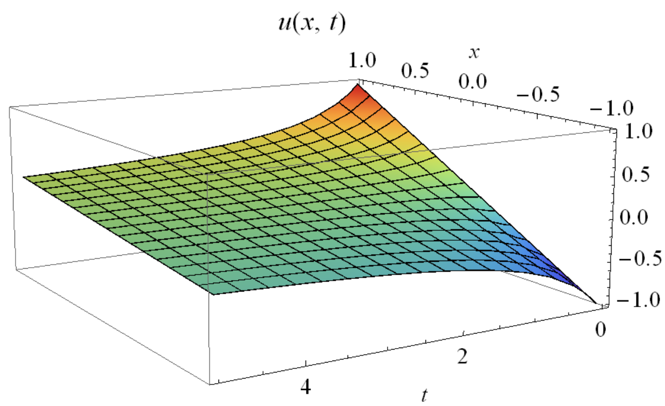

Figure 2: This is a plot of the two-dimensional solution surface $u(x,t)$ in three-dimensional space for $-5 < x < 5$ and $0 < t < 5$.

To determine where in the xt-plane the solution to the PDE is valid, it's necessary to plot the characteristic curves along with the given data curve at $t = 0$. Since $u = f(\xi) = \xi$, the equation for the characteristics becomes

$$x = ut + \xi = \xi t + \xi = \xi(t + 1).$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently. ξ is chosen here to go from -10 to 10, incrementing by 0.25 each time.

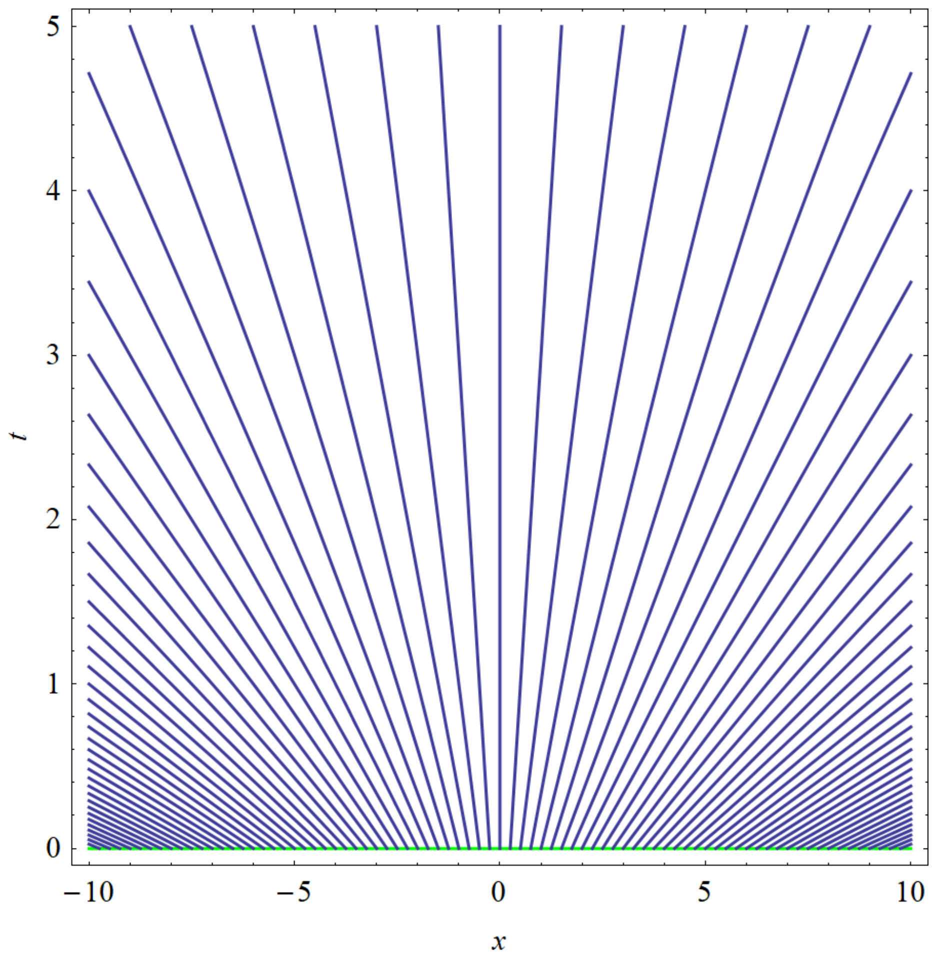

Figure 3: Plot of the characteristic curves in the xt-plane along with the given data curve ($t = 0$) in green.

There are a few things to observe here. All the characteristics are straight lines. This is because the right side of the PDE is 0. The solution to the PDE is unique and constant along each of the characteristics. The fact that none of the characteristics intersect for $t > 0$ means that no shock wave develops. This could have been foreseen from the fact that the initial condition is a monotonically increasing function of x, $u(x, 0) = x$. Finally, because each of the characteristics intersects the data curve exactly once and extends throughout the upper half-plane, the boxed solution we have for the PDE is valid for all x and $t > -1$.