Exercise 6

Verify by differentiation that the formula (12) provides a solution of the differential equation (6).

Solution

Here we have to check that

$$u(x,t) = \frac{1 + 2tx - \sqrt{1 + 4tx}}{2t^2} \tag{12}$$

is in fact a solution to the PDE

$$u_t + uu_x = 0. \tag{6}$$

To make it easier to differentiate the function, split up the fraction before anything else.

$$\begin{aligned} u(x,t) &= \frac{1}{2t^2} + \frac{x}{t} - \frac{1}{2t^2}\sqrt{1 + 4tx} \\ &= \frac{1}{2}t^{-2} + xt^{-1} - \frac{1}{2}t^{-2}\sqrt{1 + 4tx} \end{aligned}$$

Find u_t.

$$\begin{aligned} u_t &= -t^{-3} - xt^{-2} + t^{-3}\sqrt{1 + 4tx} - \frac{1}{2}t^{-2} \cdot \frac{1}{2}(1 + 4tx)^{-1/2} \cdot 4x \\ &= -t^{-3} - xt^{-2} + t^{-3}\sqrt{1 + 4tx} - xt^{-2}(1 + 4tx)^{-1/2} \end{aligned}$$

Find u_x.

$$\begin{aligned} u_x &= t^{-1} - \frac{1}{2}t^{-2} \cdot \frac{1}{2}(1 + 4tx)^{-1/2} \cdot 4t \\ &= t^{-1} - t^{-1}(1 + 4tx)^{-1/2} \end{aligned}$$

We have the following for uu_x then.

$$\begin{aligned} uu_x &= \left(\frac{1}{2}t^{-2} + xt^{-1} - \frac{1}{2}t^{-2}\sqrt{1 + 4tx}\right)\left[t^{-1} - t^{-1}(1 + 4tx)^{-1/2}\right] \\ &= \frac{1}{2}t^{-3} - \frac{1}{2}t^{-3}(1 + 4tx)^{-1/2} + xt^{-2} - xt^{-2}(1 + 4tx)^{-1/2} - \frac{1}{2}t^{-3}\sqrt{1 + 4tx} + \frac{1}{2}t^{-3} \end{aligned}$$

Now the sum can be evaluated.

$$\begin{aligned} u_t + uu_x &= -\cancel{t^{-3}} - \cancel{xt^{-2}} + t^{-3}\sqrt{1 + 4tx} - xt^{-2}(1 + 4tx)^{-1/2} \\ &\quad + \cancel{\frac{1}{2}t^{-3}} - \frac{1}{2}t^{-3}(1 + 4tx)^{-1/2} + \cancel{xt^{-2}} - xt^{-2}(1 + 4tx)^{-1/2} - \frac{1}{2}t^{-3}\sqrt{1 + 4tx} + \cancel{\frac{1}{2}t^{-3}} \\ &= -2xt^{-2}(1 + 4tx)^{-1/2} + \frac{1}{2}t^{-3}\sqrt{1 + 4tx} - \frac{1}{2}t^{-3}(1 + 4tx)^{-1/2} \\ &= t^{-3}(1 + 4tx)^{-1/2}\left[-2xt + \frac{1}{2}(1 + 4tx) - \frac{1}{2}\right] \\ &= t^{-3}(1 + 4tx)^{-1/2}\left[-\cancel{2xt} + \cancel{\frac{1}{2}} + \cancel{2tx} - \cancel{\frac{1}{2}}\right] \\ &= 0 \end{aligned}$$

Therefore, (12) is a solution of (6).