Exercise 7

Solve $xu_t + uu_x = 0$ with $u(x,0) = x$. (*Hint:* Change variables $x \mapsto x^2$.)

Solution

For a function of two variables $u = u(x,t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Divide both sides of the given PDE by x.

$$u_t + \frac{u}{x}u_x = 0$$

Comparing the right side of the previous equation with this PDE, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = \frac{u}{x}, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 0, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. Equation (2) tells us that u is independent of t, so the characteristic curves can be obtained by applying the method of separation of variables to equation (1).

$$x\,dx = u\,dt$$

Integrate both sides.

$$\frac{x^2}{2} = ut + C$$

Multiply both sides by 2 and let ξ be the new constant.

$$x^2 = 2ut + \xi$$

Solve this equation for ξ.

$$\xi = x^2 - 2ut$$

The general solution to the PDE is then

$$u(x,t) = f(x^2 - 2ut).$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x,0) = f(x^2) = x$$

Let $w = x^2$. Then $f(w) = \sqrt{w}$. Any expression can be used for w, including $x^2 - 2ut$.

$$f(x^2 - 2ut) = \sqrt{x^2 - 2ut}$$

Thus,

$$u(x,t) = \sqrt{x^2 - 2ut}.$$

Solve this equation for u.

$$u = \sqrt{x^2 - 2ut}$$
$$u^2 = x^2 - 2ut$$

$$u^2 + 2tu - x^2 = 0$$

Use the quadratic formula.

$$u = \frac{-2t \pm \sqrt{4t^2 + 4x^2}}{2}$$
$$u = -t \pm \sqrt{t^2 + x^2}$$

Note that the square root of x^2 is $|x|$. In order to satisfy the initial condition, $u(x,0) = x$, we choose the plus sign for $x > 0$ and the minus sign for $x < 0$. Therefore,

$$\boxed{u(x,t) = \begin{cases} -t + \sqrt{t^2 + x^2}, & x > 0 \\ -t - \sqrt{t^2 + x^2}, & x < 0 \end{cases}.}$$

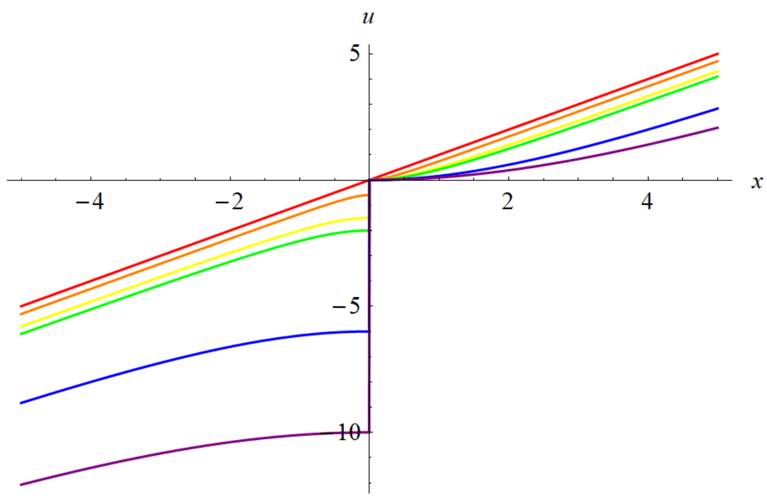

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.3$, $t = 0.75$, $t = 1$, $t = 3$, $t = 5$, respectively.

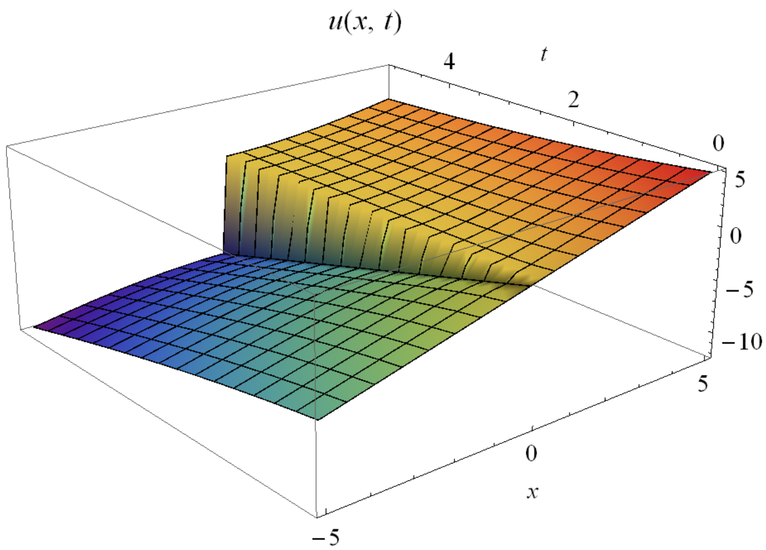

Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional space for $-5 < x < 5$ and $0 < t < 5$.

To determine where in the xt-plane the solution to the PDE is valid, it's necessary to plot the characteristic curves along with the given data curve at $t = 0$. It was determined earlier that $u = f(\xi) = \sqrt{\xi}$, so the equation for the characteristics becomes

$$x^2 = 2\sqrt{\xi}t + \xi.$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently.

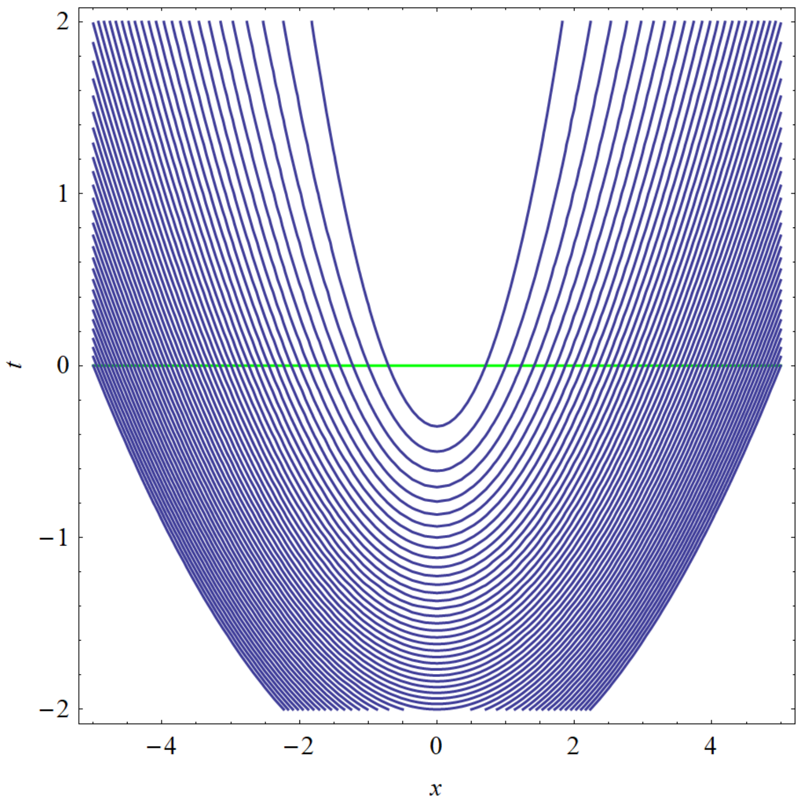

Figure 3: Plot of the characteristic curves in the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -25 to 25, incrementing by 0.5 each time.

u is constant along each of the characteristics. Since none of the characteristics cross, there are no shock waves. At first glance, it might seem there's a problem since the data curve intersects each characteristic curve twice. This is all right, though, because u is the same at x as it is at $-x$ as can be seen from the solution. The slope dx/dt is also consistent at each intersection because it equals u/x; for $x < 0$ the slope will be negative, and for $x > 0$ the slope will be positive. Because the characteristic curves cover the entire xt-plane and the data curve goes through all of them, the boxed solution is valid for all x and t.