Exercise 2

Solve $u_{tt} = c^2 u_{xx},\ u(x,0) = \log(1+x^2),\ u_t(x,0) = 4 + x.$

Solution

Solution by Operator Factorization

By factoring the wave equation and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by bringing all terms of the PDE to one side.

$$u_{tt} - c^2 u_{xx} = 0$$

Write the left side as an operator acting on u, $\mathscr{L}u$.

$$(\partial_t^2 - c^2\partial_x^2)u = 0$$

Now factor the operator.

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

If we let $v = (\partial_t + c\partial_x)u$, then what remains is $(\partial_t - c\partial_x)v = 0$. That is,

$$\begin{cases} u_t + cu_x = v \\ v_t - cv_x = 0 \end{cases}.$$

On the paths defined by

$$\frac{dx}{dt} = -c, \quad x(\xi, 0) = \xi \tag{1}$$

the PDE on the bottom reduces to an ODE,

$$\frac{dv}{dt} = 0. \tag{2}$$

That is, $v = v(x,t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$v(\xi, t) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by integration gives

$$x = -ct + \xi.$$

Solve now for ξ.

$$\xi = x + ct$$

Therefore,

$$v(x,t) = f(x+ct).$$

We can check that this is the solution of the bottom PDE.

$$v_t = cf'$$
$$v_x = f'$$

$v_t - cv_x = 0$, so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

$$u_t + cu_x = f(x + ct).$$

On the paths defined by

$$\frac{dx}{dt} = c, \quad x(\eta, 0) = \eta \tag{3}$$

the PDE on top reduces to an ODE,

$$\frac{du}{dt} = f(x + ct). \tag{4}$$

That is, $u = u(x, t)$ is constant on the characteristics defined by (3). Solving (3) by integration, we find that

$$x = ct + \eta.$$

Solve now for η.

$$\eta = x - ct$$

Now that we know x in terms of η, we can solve (4) for u.

$$\frac{du}{dt} = f(2ct + \eta)$$

Integrate both sides now with respect to t.

$$u(\eta, t) = \int^t f(2cs + \eta)\, ds + g(\eta),$$

where g is an arbitrary function of η. Now that the integration is done, change back to the original variables.

$$u(x, t) = \int^t f(2cs + x - ct)\, ds + g(x - ct)$$

To solve the integral, use a substitution. Plug in $s = t$ to get the upper limit of integration.

$$\begin{gathered} w = 2cs + x - ct \\ dw = 2c\, ds \quad \rightarrow \quad \frac{1}{2c}\, dw = ds \end{gathered}$$

It becomes

$$\begin{gathered} u(x, t) = \int^{x+ct} f(w) \frac{1}{2c}\, dw + g(x - ct) \\ u(x, t) = \frac{1}{2c} F(x + ct) + g(x - ct). \end{gathered}$$

Now that we have the general solution for $u(x, t)$, we use the initial conditions to determine the unknown functions, F and g.

$$\begin{aligned} u(x, 0) = \log(1 + x^2) \quad &\rightarrow \quad \frac{1}{2c} F(x) + g(x) = \log(1 + x^2) \\ u_t(x, 0) = 4 + x \quad &\rightarrow \quad \frac{1}{2} F'(x) - cg'(x) = 4 + x \end{aligned}$$

Multiply both sides of the top equation by c and differentiate both sides with respect to x.

$$\frac{1}{2}F'(x) + cg'(x) = c\frac{2x}{1+x^2}$$
$$\frac{1}{2}F'(x) - cg'(x) = 4 + x$$

Adding these two equations gives us

$$F'(x) = c\frac{2x}{1+x^2} + 4 + x.$$

Subtracting the two equations gives us

$$2cg'(x) = c\frac{2x}{1+x^2} - (4+x).$$

Divide both sides by $2c$ to isolate g'.

$$g'(x) = \frac{x}{1+x^2} - \frac{1}{2c}(4+x)$$

Solve for F and g by integrating both sides.

$$F(x) = c\log(1+x^2) + \frac{1}{2}x(x+8)$$
$$g(x) = \frac{1}{2}\log(1+x^2) - \frac{1}{4c}x(x+8)$$

What we solved for are actually $F(w)$ and $g(w)$, where w is any expression. Thus,

$$F(x+ct) = c\log[1+(x+ct)^2] + \frac{1}{2}(x+ct)[(x+ct)+8]$$
$$g(x-ct) = \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}(x-ct)[(x-ct)+8].$$

Plugging these into $u(x,t)$, we obtain the answer to the initial value problem.

$$u(x,t) = \frac{1}{2c}\left\{c\log[1+(x+ct)^2] + \frac{1}{2}(x+ct)[(x+ct)+8]\right\} + \left\{\frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}(x-ct)[(x-ct)+8]\right\}$$

$$u(x,t) = \frac{1}{2}\left\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\right\} + \frac{1}{4c}\left\{(x+ct)[(x+ct)+8] - (x-ct)[(x-ct)+8]\right\}$$

$$u(x,t) = \frac{1}{2}\log[1+(x+ct)^2][1+(x-ct)^2] + \frac{1}{4c}[4ct(x+4)]$$

Therefore,

$$u(x,t) = \log\sqrt{[1+(x+ct)^2][1+(x-ct)^2]} + t(x+4).$$

Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dx}{dt} &= \frac{1}{2}\left(\pm\sqrt{0 + 4c^2}\right)\\ \frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xt-plane.

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solving for the constants of integration,

$$\begin{aligned}C_1 &= x - ct = \phi(x,t)\\ C_2 &= x + ct = \psi(x,t).\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x,t) = x - ct$ and $\eta = \psi(x,t) = x + ct$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule,

$$\begin{aligned}A^* &= A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2\\ B^* &= 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x\\ C^* &= A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2\\ D^* &= A\xi_{tt} + B\xi_{xt} + C\xi_{xx} + D\xi_t + E\xi_x\\ E^* &= A\eta_{tt} + B\eta_{xt} + C\eta_{xx} + D\eta_t + E\eta_x\\ F^* &= F\\ G^* &= G.\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -4c^2$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-4c^2u_{\xi\eta} = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = 0.$$

This is known as the first canonical form of the wave equation. We can solve it by integrating both sides with respect to η and then integrating both sides again with respect to ξ.

$$u_\xi = f(\xi),$$

where f is an arbitrary function of ξ.

$$u(\xi,\eta) = F(\xi) + G(\eta),$$

where F and G are arbitrary functions of ξ and η, respectively. Now change back to the original variables with the substitutions, $\xi = x - ct$ and $\eta = x + ct$, to get the general solution of the wave equation.

$$u(x,t) = F(x - ct) + G(x + ct)$$

We determine these unknown functions by using the initial conditions.

$$\begin{aligned} u(x,0) = \log(1+x^2) \quad &\rightarrow \quad F(x) + G(x) = \log(1+x^2) \\ u_t(x,0) = 4 + x \quad &\rightarrow \quad -cF'(x) + cG'(x) = 4 + x \end{aligned}$$

Multiply both sides of the top equation by c and differentiate both sides with respect to x.

$$\begin{aligned} cF'(x) + cG'(x) &= c\frac{2x}{1+x^2} \\ -cF'(x) + cG'(x) &= 4 + x \end{aligned}$$

Adding these two equations gives us

$$2cG'(x) = c\frac{2x}{1+x^2} + 4 + x.$$

Subtracting the two equations gives us

$$2cF'(x) = c\frac{2x}{1+x^2} - (4 + x).$$

Divide both sides by $2c$ to isolate F' and G'.

$$\begin{aligned} F'(x) &= \frac{x}{1+x^2} - \frac{1}{2c}(4+x) \\ G'(x) &= \frac{x}{1+x^2} + \frac{1}{2c}(4+x) \end{aligned}$$

Solve for F and G by integrating both sides.

$$\begin{aligned} F(x) &= \frac{1}{2}\log(1+x^2) + \frac{1}{4c}x(x+8) \\ G(x) &= \frac{1}{2}\log(1+x^2) - \frac{1}{4c}x(x+8) \end{aligned}$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression. Thus,

$$\begin{aligned} F(x+ct) &= \frac{1}{2}\log[1+(x+ct)^2] + \frac{1}{4c}(x+ct)[(x+ct)+8] \\ G(x-ct) &= \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}(x-ct)[(x-ct)+8]. \end{aligned}$$

Plugging these into $u(x,t)$, we obtain the answer to the initial value problem.

$$\begin{aligned} u(x,t) = \frac{1}{2}\log[1+(x+ct)^2] &+ \frac{1}{4c}(x+ct)[(x+ct)+8] \\ &+ \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}(x-ct)[(x-ct)+8] \end{aligned}$$

$$u(x,t) = \frac{1}{2}\left\{\log[1+(x+ct)^2] + \log[1+(x-ct)^2]\right\} + \frac{1}{4c}\left\{(x+ct)[(x+ct)+8] - (x-ct)[(x-ct)+8]\right\}$$

$$u(x,t) = \frac{1}{2}\log[1+(x+ct)^2][1+(x-ct)^2] + \frac{1}{4c}[4ct(x+4)]$$

Therefore,

$$u(x,t) = \log\sqrt{[1+(x+ct)^2][1+(x-ct)^2]} + t(x+4).$$

This is the same answer that we obtained with operator factorization. Checking that this is the solution to the wave equation is quite involved. To speed up the task, we will show that the general solution, $u(x,t) = F(x-ct) + G(x+ct)$, satisfies it. The solution we obtained is just a special case of this that satisfies the given initial conditions.

$$\begin{aligned} u_t &= -cF'(x-ct) + cG'(x+ct) \\ u_{tt} &= c^2F''(x-ct) + c^2G''(x+ct) \\ u_x &= F'(x-ct) + G'(x+ct) \\ u_{xx} &= F''(x-ct) + G''(x+ct) \end{aligned}$$

$u_{tt} = c^2u_{xx}$, so this is the solution to the wave equation. Now check the initial conditions.

$$\begin{aligned} u(x,0) &= \log\sqrt{(1+x^2)^2} = \log(1+x^2) \\ u_t(x,0) &= 4 + x + \left.\frac{2c^2t(1+c^2t^2-x^2)}{c^4t^4 - 2c^2t^2(x^2-1) + (x^2+1)^2}\right|_{t=0} = 4 + x \end{aligned}$$

To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath's "Linear Partial Differential Equations for Scientists and Engineers, 4th Edition."