Exercise 4

Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x+ct)+g(x-ct)$.

Solution

Solution by Operator Factorization

By factoring the wave equation and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by bringing all terms of the PDE to one side.

$$u_{tt} - c^2 u_{xx} = 0$$

Write the left side as an operator acting on u, $\mathscr{L}u$.

$$(\partial_t^2 - c^2\partial_x^2)u = 0$$

Now factor the operator.

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

If we let $v = (\partial_t + c\partial_x)u$, then what remains is $(\partial_t - c\partial_x)v = 0$. That is,

$$\begin{cases} u_t + cu_x = v \\ v_t - cv_x = 0 \end{cases}.$$

On the paths defined by

$$\frac{dx}{dt} = -c, \quad x(\xi, 0) = \xi \tag{1}$$

the PDE on the bottom reduces to an ODE,

$$\frac{dv}{dt} = 0. \tag{2}$$

That is, $v = v(x,t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$v(\xi, t) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by integration gives

$$x = -ct + \xi$$

Solve now for ξ.

$$\xi = x + ct$$

Therefore,

$$v(x,t) = f(x+ct).$$

We can check that this is the solution of the bottom PDE.

$$v_t = cf'$$
$$v_x = f'$$

$v_t - cv_x = 0$, so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

$$u_t + cu_x = f(x + ct).$$

On the paths defined by

$$\frac{dx}{dt} = c, \quad x(\eta, 0) = \eta \tag{3}$$

the PDE on top reduces to an ODE,

$$\frac{du}{dt} = f(x + ct). \tag{4}$$

That is, $u = u(x,t)$ is constant on the characteristics defined by (3). Solving (3) by integration, we find that

$$x = ct + \eta.$$

Solve now for η.

$$\eta = x - ct$$

Now that we know x in terms of η, we can solve (4) for u.

$$\frac{du}{dt} = f(2ct + \eta)$$

Integrate both sides now with respect to t.

$$u(\eta, t) = \int^t f(2cs + \eta)\, ds + g(\eta),$$

where g is an arbitrary function of η. Now that the integration is done, change back to the original variables.

$$u(x,t) = \int^t f(2cs + x - ct)\, ds + g(x - ct)$$

To solve the integral, use a substitution. Plug in $s = t$ to get the upper limit of integration.

$$w = 2cs + x - ct$$
$$dw = 2c\, ds \quad \to \quad \frac{1}{2c}\, dw = ds$$

It becomes

$$u(x,t) = \int^{x+ct} f(w) \frac{1}{2c}\, dw + g(x - ct)$$
$$u(x,t) = \frac{1}{2c} F(x + ct) + g(x - ct).$$

To get rid of the factor in front of F, introduce a new arbitrary function, H.

$$u(x,t) = H(x + ct) + g(x - ct)$$

Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dx}{dt} &= \frac{1}{2}\left(\pm\sqrt{0 + 4c^2}\right) \\
\frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xt-plane.

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solving for the constants of integration,

$$\begin{aligned}
C_1 &= x - ct = \phi(x,t) \\
C_2 &= x + ct = \psi(x,t).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x,t) = x - ct$ and $\eta = \psi(x,t) = x + ct$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule,

$$\begin{aligned}
A^* &= A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2 \\
B^* &= 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x \\
C^* &= A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2 \\
D^* &= A\xi_{tt} + B\xi_{xt} + C\xi_{xx} + D\xi_t + E\xi_x \\
E^* &= A\eta_{tt} + B\eta_{xt} + C\eta_{xx} + D\eta_t + E\eta_x \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -4c^2$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-4c^2 u_{\xi\eta} = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = 0.$$

This is known as the first canonical form of the wave equation. We can solve it by integrating both sides with respect to η and then integrating both sides again with respect to ξ.

$$u_\xi = f(\xi),$$

where f is an arbitrary function of ξ.

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

where F and G are arbitrary functions of ξ and η, respectively. Now change back to the original variables with the substitutions, $\xi = x - ct$ and $\eta = x + ct$, to get the general solution of the wave equation.

$$u(x, t) = F(x - ct) + G(x + ct)$$

This is the same answer that we obtained with operator factorization. We can check that this is the solution to the wave equation.

$$\begin{aligned}
u_t &= -cF'(x - ct) + cG'(x + ct) \\
u_{tt} &= c^2 F''(x - ct) + c^2 G''(x + ct) \\
u_x &= F'(x - ct) + G'(x + ct) \\
u_{xx} &= F''(x - ct) + G''(x + ct)
\end{aligned}$$

$u_{tt} = c^2 u_{xx}$, so this is the solution to the wave equation. To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath's "Linear Partial Differential Equations for Scientists and Engineers, 4th Edition."