Exercise 5

(*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c$, a/c, $3a/2c$, $2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\{\text{length of } (x-ct,\ x+ct) \cap (-a,a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2, x + a/2) \cap (-a, a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

Solution

The goal for this exercise is to plot the solution to the wave equation with initial conditions versus x at five instants in time:

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}.$$

The solution to the wave equation with initial conditions, $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$, is given by D'Alembert's formula,

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

Since $\phi(x) = 0$, the solution reduces to

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

Following the hint, we will set t equal to each of the five values and then determine

$$\frac{1}{2c}\{\text{length of } (x-ct,\ x+ct) \cap (-a,a)\}$$

for various intervals of x in order to evaluate $u(x, t)$. The reason we only care about the length from $x - ct$ to $x + ct$ here is because the integrand $\psi(x)$ is equal to 1 from $x = -a$ to $x = a$ and 0 everywhere else.

The String Profile for $t = \frac{a}{2c}$

Setting $t = \frac{a}{2c}$, we find that $x \pm ct = x \pm \frac{a}{2}$. The first interval of x we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + \frac{a}{2}$, touches the left end of the interval, $(-a, a)$, at the highest value of x.

$$\text{For } x < -\frac{3a}{2}, \quad \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \cap (-a,a) = (-\infty, -a) \cap (-a,a) = \emptyset$$

$$\rightarrow \quad u\left(x, \frac{a}{2c}\right) = \frac{1}{2c}\int_{-a}^{-a} 1\,ds = 0$$

The next interval of x we consider is the one where the right-most point, $x + \frac{a}{2}$, is inside the interval, $(-a, a)$, but the left-most point, $x - \frac{a}{2}$, is not. The distance from $x - \frac{a}{2}$ to $x + \frac{a}{2}$ is a, so the next interval of x will be from where we left off, $-\frac{3a}{2}$, to $-\frac{a}{2}$, where the left-most point touches the left end of the interval.

$$\begin{aligned} \text{For } -\frac{3a}{2} < x < -\frac{a}{2}, \quad & \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \cap (-a, a) = \left(-a, x + \frac{a}{2}\right) \\ \rightarrow \quad & u\left(x, \frac{a}{2c}\right) = \frac{1}{2c}\int_{-a}^{x+\frac{a}{2}} 1\, ds = \frac{1}{2c}\left(x + \frac{3a}{2}\right) \end{aligned}$$

The next interval of x we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are inside the interval, $(-a, a)$. This interval of x will start from where we left off, $-\frac{a}{2}$, and go to $\frac{a}{2}$. The left-most point, $x - \frac{a}{2}$, touches the left end of the interval, $(-a, a)$, at the lowest value of x, and the right-most point, $x + \frac{a}{2}$, touches the right end of the interval, $(-a, a)$, at the highest value of x.

$$\begin{aligned} \text{For } -\frac{a}{2} < x < \frac{a}{2}, \quad & \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \cap (-a, a) = \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \\ \rightarrow \quad & u\left(x, \frac{a}{2c}\right) = \frac{1}{2c}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} 1\, ds = \frac{a}{2c} \end{aligned}$$

The next interval of x we consider is the one where the right-most point, $x + \frac{a}{2}$, is outside the right end of the interval, $(-a, a)$, and the left-most point, $x - \frac{a}{2}$, is still inside of it. This interval of x will start from where we left off, $\frac{a}{2}$, and go to $\frac{3a}{2}$, where the left-most point touches the right end of the interval, $(-a, a)$.

$$\begin{aligned} \text{For } \frac{a}{2} < x < \frac{3a}{2}, \quad & \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \cap (-a, a) = \left(x - \frac{a}{2}, a\right) \\ \rightarrow \quad & u\left(x, \frac{a}{2c}\right) = \frac{1}{2c}\int_{x-\frac{a}{2}}^{a} 1\, ds = \frac{1}{2c}\left(\frac{3a}{2} - x\right) \end{aligned}$$

The last interval of x we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are outside the interval, $(-a, a)$, on the right side. The interval of x will start from where we left off, $\frac{3a}{2}$, and go to ∞. The left-most point, $x - \frac{a}{2}$, touches the right end of the interval, $(-a, a)$, at the lowest value of x.

$$\begin{aligned} \text{For } x > \frac{3a}{2}, \quad & \left(x - \frac{a}{2}, x + \frac{a}{2}\right) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset \\ \rightarrow \quad & u\left(x, \frac{a}{2c}\right) = \int_{a}^{a} 1\, ds = 0 \end{aligned}$$

Now that we know $u\left(x, \frac{a}{2c}\right)$ for all x, it can be plotted.

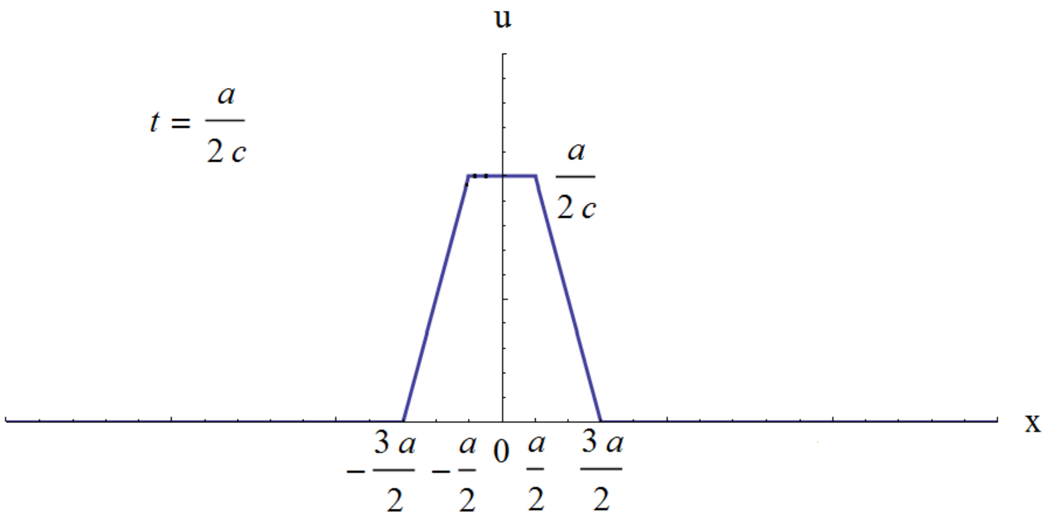

Figure 1: Plot of $u(x,t)$ when $t = \frac{a}{2c}$.

The String Profile for $t = \frac{a}{c}$

Setting $t = \frac{a}{c}$, we find that $x \pm ct = x \pm a$. The first interval of x we consider is the one where the left and right-most points, $x \pm a$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + a$, touches the left end of the interval, $(-a, a)$, at the highest value of x.

$$\text{For } x < -2a, \quad (x-a, x+a) \cap (-a,a) = (-\infty, -a) \cap (-a, a) = \emptyset$$
$$\to \quad u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\int_{-a}^{-a} 1\,ds = 0$$

The next interval of x we consider is the one where the right-most point, $x + a$, is inside the interval, $(-a, a)$, but the left-most point, $x - a$, is not. The distance from $x - a$ to $x + a$ is $2a$, so the next interval of x will be from where we left off, $-2a$, to 0.

$$\text{For } -2a < x < 0, \quad (x-a, x+a) \cap (-a,a) = (-a, x+a)$$
$$\to \quad u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\int_{-a}^{x+a} 1\,ds = \frac{1}{2c}(x+2a)$$

Since the interval, $(-a, a)$, is $2a$ units long as well, the next interval of x we consider is the one where the right-most point, $x + a$, is outside the interval, $(-a, a)$, and the left-most point, $x - a$, is still inside it. This interval of x will start from where we left off, 0, and go to $2a$, where the left-most point touches the right end of the interval, $(-a, a)$.

$$\text{For } 0 < x < 2a, \quad (x-a, x+a) \cap (-a,a) = (x-a, a)$$
$$\to \quad u\left(x, \frac{a}{c}\right) = \frac{1}{2c}\int_{x-a}^{a} 1\,ds = \frac{1}{2c}(2a-x)$$

The last interval of x we consider is the one where the left and right-most points, $x \pm a$, are outside the interval $(-a, a)$ on the right side. The interval of x will start from where we left off, $2a$, and go to ∞. The left-most point, $x - a$, touches the right end of the interval, $(-a, a)$, at the lowest value of x.

$$\text{For } x > 2a, \quad (x-a, x+a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{a}{c}\right) = \int_a^a 1\,ds = 0$$

Now that we know $u\left(x, \frac{a}{c}\right)$ for all x, it can be plotted.

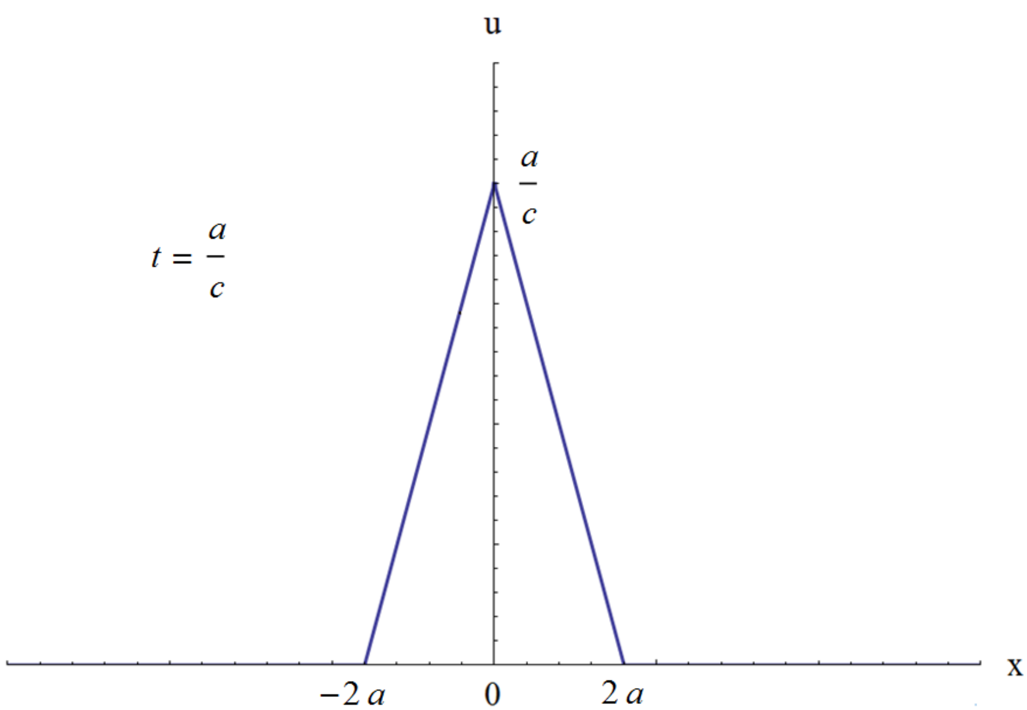

Figure 2: Plot of $u(x,t)$ when $t = \frac{a}{c}$.

The String Profile for $t = \frac{3a}{2c}$

Setting $t = \frac{3a}{2c}$, we find that $x \pm ct = x \pm \frac{3a}{2}$. The first interval of x we consider is the one where the left and right-most points, $x \pm \frac{3a}{2}$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + \frac{3a}{2}$, touches the left end of the interval $(-a, a)$ at the highest value of x.

$$\text{For } x < -\frac{5a}{2}, \quad \left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{-a}^{-a} 1\,ds = 0$$

The next interval of x we consider is the one where the right-most point, $x + \frac{3a}{2}$, is inside the interval, $(-a, a)$, but the left-most point, $x - \frac{3a}{2}$, is not. The distance from $x - \frac{3a}{2}$ to $x + \frac{3a}{2}$ is $3a$, which is bigger than the interval, (-a,a). This means the next interval of x will be from where we left off, $-\frac{5a}{2}$, to $-\frac{a}{2}$, where the right-most point, $x + \frac{3a}{2}$, touches the right end of the interval, $(-a, a)$.

$$\begin{aligned} \text{For } -\frac{5a}{2} < x < -\frac{a}{2}, \quad & \left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) = \left(-a, x + \frac{3a}{2}\right) \\ & \rightarrow \quad u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{-a}^{x+\frac{3a}{2}} 1\, ds = \frac{1}{2c}\left(x + \frac{5a}{2}\right) \end{aligned}$$

Since the interval, $(-a, a)$, is only $2a$ units long, the next interval of x we consider is the one where the right-most point, $x + \frac{3a}{2}$, is outside the interval, $(-a, a)$, on the right, and the left-most point, $x - \frac{3a}{2}$, is still outside the left end. This interval of x will start from where we left off, $-\frac{a}{2}$, and go to $\frac{a}{2}$, where the left-most point touches the left end of the interval, $(-a, a)$.

$$\begin{aligned} \text{For } -\frac{a}{2} < x < \frac{a}{2}, \quad & \left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) = (-a, a) \\ & \rightarrow \quad u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{-a}^{a} 1\, ds = \frac{a}{c} \end{aligned}$$

The next interval we consider is the one where the left-most point, $x - \frac{3a}{2}$, is inside the left end of the interval, $(-a, a)$, and the right-most point, $x + \frac{3a}{2}$, is outside $(-a, a)$ on the right side. The interval of x will start from where we left off, $\frac{a}{2}$, and go to $\frac{5a}{2}$, where the left-most point touches the right end of $(-a, a)$.

$$\begin{aligned} \text{For } \frac{a}{2} < x < \frac{5a}{2}, \quad & \left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) = \left(x - \frac{3a}{2}, a\right) \\ & \rightarrow \quad u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{x-\frac{3a}{2}}^{a} 1\, ds = \frac{1}{2c}\left(\frac{5a}{2} - x\right) \end{aligned}$$

The last interval of x we consider is the one where the left and right-most points, $x \pm \frac{3a}{2}$, are outside the interval, $(-a, a)$, on the right side. The interval of x will start from where we left off, $\frac{5a}{2}$, and go to ∞. The left-most point, $x - \frac{3a}{2}$, touches the right end of the interval, $(-a, a)$, at the lowest value of x.

$$\begin{aligned} \text{For } x > \frac{5a}{2}, \quad & \left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset \\ & \rightarrow \quad u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\int_{a}^{a} 1\, ds = 0 \end{aligned}$$

Now that we know $u\left(x, \frac{3a}{2c}\right)$ for all x, we can plot it.

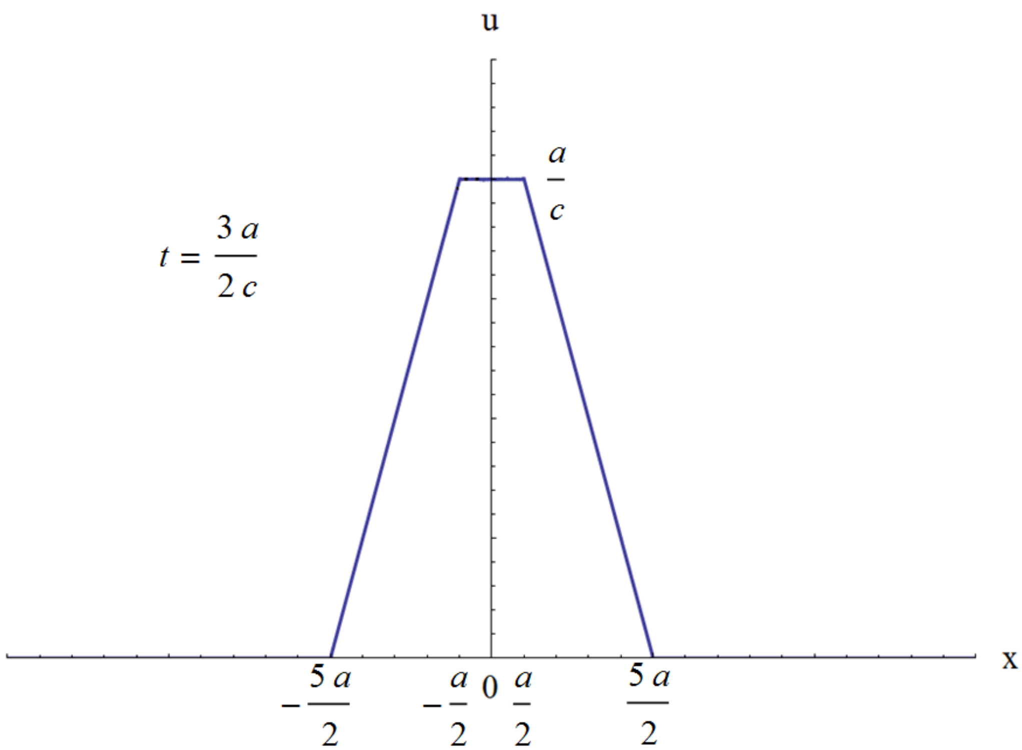

Figure 3: Plot of $u(x,t)$ when $t = \frac{3a}{2c}$.

The String Profile for $t = \frac{2a}{c}$

Setting $t = \frac{2a}{c}$, we find that $x \pm ct = x \pm 2a$. The first interval of x we consider is the one where the left and right-most points, $x \pm 2a$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + 2a$, touches the left end of the interval, $(-a, a)$, at the highest value of x.

$$\text{For } x < -3a, \quad (x-2a, x+2a) \cap (-a,a) = (-\infty, -a) \cap (-a,a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{-a}^{-a} 1\, ds = 0$$

The next interval of x we consider is the one where the right-most point, $x + 2a$, is inside the interval, $(-a, a)$, but the left-most point, $x - 2a$, is not. The distance from $x - 2a$ to $x + 2a$ is $4a$, which is bigger than the interval, (-a,a). This means the next interval of x will be from where we left off, $-3a$, to $-a$, where the right-most point, $x + 2a$, touches the right end of the interval, $(-a, a)$.

$$\text{For } -3a < x < -a, \quad (x-2a, x+2a) \cap (-a,a) = (-a, x+2a)$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{-a}^{x+2a} 1\, ds = \frac{1}{2c}(x+3a)$$

Since the interval $(-a, a)$ is only $2a$ units long, the next interval of x we consider is the one where the right-most point, $x + 2a$, is outside the interval, $(-a, a)$, on the right, and the left-most point, $x - 2a$, is still outside the left end. This interval of x will start from where we left off, $-a$, and go to a, where the left-most point touches the left end of the interval, $(-a, a)$.

$$\text{For } -a < x < a, \quad (x - 2a, x + 2a) \cap (-a, a) = (-a, a)$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{-a}^{a} 1\, ds = \frac{a}{c}$$

The next interval of x we consider is the one where the left-most point, $x - 2a$, is inside the interval, $(-a, a)$, and the right-most point, $x + \frac{3a}{2}$, is outside the interval, $(-a, a)$, on the right side. This interval of x will start from where we left off, a, and go until $3a$, where the left-most point touches the right end of $(-a, a)$.

$$\text{For } a < x < 3a, \quad (x - 2a, x + 2a) \cap (-a, a) = (x - 2a, a)$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{x-2a}^{a} 1\, ds = \frac{1}{2c}(3a - x)$$

The last interval of x we consider is the one where the left and right-most points, $x \pm 2a$, are outside the interval, $(-a, a)$, on the right side. This interval of x will start from where we left off, $3a$, and go to ∞. The left-most point, $x - 2a$, touches the right end of the interval, $(-a, a)$, at the lowest value of x.

$$\text{For } x > 3a, \quad (x - 2a, x + 2a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\int_{a}^{a} 1\, ds = 0$$

Now that we know $u\left(x, \frac{2a}{c}\right)$ for all x, we can plot it.

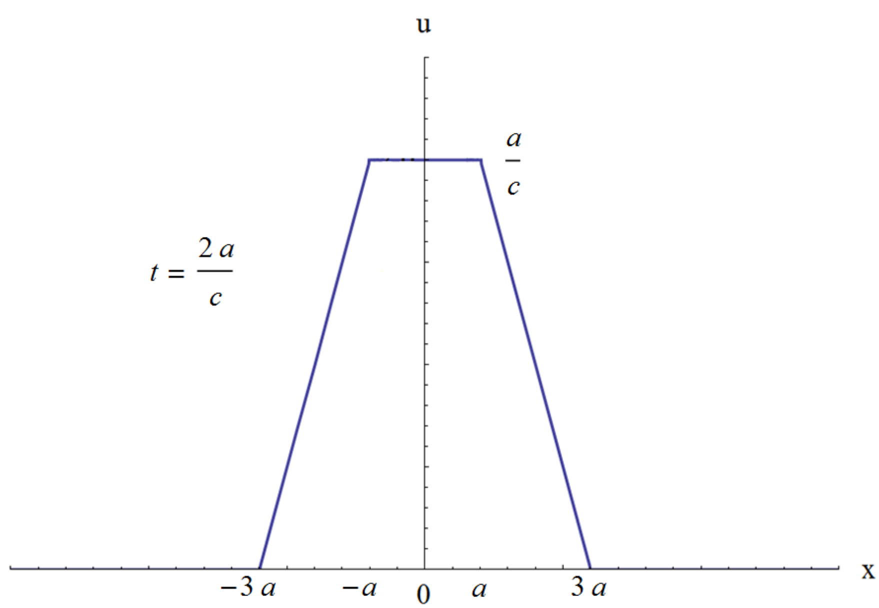

Figure 4: Plot of $u(x, t)$ when $t = \frac{2a}{c}$.

The String Profile for $t = \frac{5a}{c}$

Setting $t = \frac{5a}{c}$, we find that $x \pm ct = x \pm 5a$. The first interval of x we consider is the one where the left and right-most points, $x \pm 5a$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + 5a$, touches the left end of the interval, $(-a, a)$, at the highest value of x.

$$\text{For } x < -6a, \quad (x - 5a, x + 5a) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{-a} 1\, ds = 0$$

The next interval of x we consider is the one where the right-most point, $x + 5a$, is inside the interval, $(-a, a)$, but the left-most point, $x - 5a$, is not. The distance from $x - 5a$ to $x + 5a$ is $10a$, which is bigger than the interval, (-a,a). This means the next interval of x will be from where we left off, $-6a$, to $-4a$, where the right-most point, $x + 5a$, touches the right end of $(-a, a)$.

$$\text{For } -6a < x < -4a, \quad (x - 5a, x + 5a) \cap (-a, a) = (-a, x + 5a)$$
$$\rightarrow \quad u\left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{x+5a} 1\, ds = \frac{1}{2c}(x + 6a)$$

Since the interval $(-a, a)$ is only $2a$ units long, the next interval of x we consider is the one where the right-most point, $x + 5a$, is outside the interval, $(-a, a)$, on the right and the left-most point, $x - 5a$, is still outside the left end. This interval of x will start from where we left off, $-4a$, and go to $4a$, where the left-most point, $x - 5a$, touches the left end of the interval, $(-a, a)$.

$$\text{For } -4a < x < 4a, \quad (x - 5a, x + 5a) \cap (-a, a) = (-a, a)$$
$$\rightarrow \quad u\left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{a} 1\, ds = \frac{a}{c}$$

The next interval we consider is the one where the left-most point, $x - 5a$, is inside the interval, $(-a, a)$, and the right-most point, $x + 5a$, is outside $(-a, a)$ on the right side. The interval of x will start from where we left off, $4a$, and go to $6a$, where the left-most point, $x - 5a$, touches the right end of the interval, $(-a, a)$.

$$\text{For } 4a < x < 6a, \quad (x - 5a, x + 5a) \cap (-a, a) = (x - 5a, a)$$
$$\rightarrow \quad u\left(x, \frac{2a}{c}\right) = \frac{1}{2c} \int_{x-5a}^{a} 1\, ds = \frac{1}{2c}(6a - x)$$

The last interval of x we consider is the one where the left and right-most points, $x \pm 5a$, are outside the interval, $(-a, a)$, on the right side. The interval of x will start from where we left off, $6a$, and go to ∞. The left-most point, $x - 5a$, touches the right end of the interval, $(-a, a)$, at the lowest value of x.

$$\text{For } x > 6a, \quad (x - 5a, x + 5a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset$$
$$\rightarrow \quad u\left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{a}^{a} 1\, ds = 0$$

Now that we know $u\left(x, \frac{5a}{c}\right)$ for all x, we can plot it.

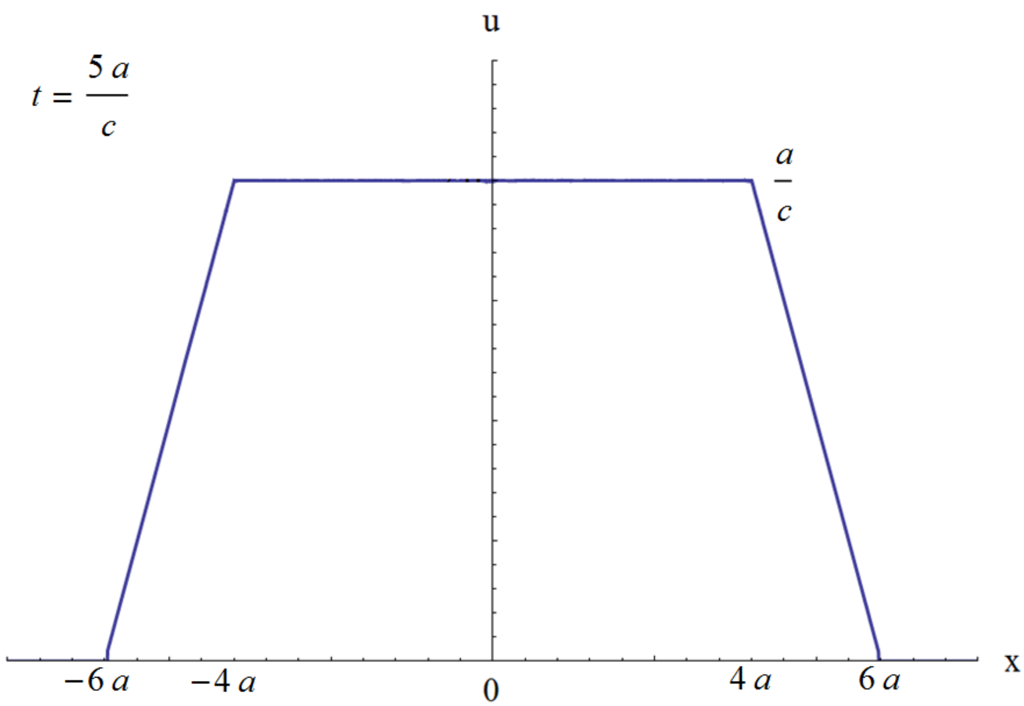

Figure 5: Plot of $u(x,t)$ when $t = \frac{5a}{c}$.