Exercise 6

In Exercise 5, find the greatest displacement, $\max_x u(x,t)$, as a function of t.

Solution

Note that Exercise 5 is called, "The hammer blow." The string is at rest with zero displacement up until $t=0$ when the hammer strikes. Thus, we will only be looking for positive values of t in this exercise. As can be seen from the graphs of the solution in Exercise 5, the greatest displacement of the string, $\max_x u(x,t)$, occurs at $x=0$ for all $t>0$. The solution of the wave equation was written there as

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds,$$

where $\psi(x)$ was the given initial condition,

$$\psi(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}.$$

If we plug in $x=0$ to this, we get

$$u(0,t) = \frac{1}{2c}\int_{-ct}^{ct} \psi(s)\,ds.$$

This integral has different values depending on what t is. If $|ct| < a$ (we can drop the absolute value bars since t is positive), then we can use 1 for $\psi(x)$.

$$\frac{1}{2c}\int_{-ct}^{ct} 1\,ds = t \quad \text{for} \quad 0 < t < \frac{a}{c} \tag{1}$$

If $|ct| \geq a$ (we can drop the absolute value bars since t is positive), then $-ct \leq -a$ after multiplying both sides by -1, and we have the following for the integral.

$$\frac{1}{2c}\int_{-ct}^{ct} \psi(s)\,ds = \frac{1}{2c}\left[\int_{-ct}^{-a} 0\,ds + \int_{-a}^{a} 1\,ds + \int_{a}^{ct} 0\,ds\right] = \frac{a}{c} \quad \text{for} \quad t \geq \frac{a}{c} \tag{2}$$

Putting (1) and (2) together, we have

$$\frac{1}{2c}\int_{-ct}^{ct} \psi(s)\,ds = \begin{cases} t & 0 < t < \frac{a}{c} \\ \frac{a}{c} & t \geq \frac{a}{c} \end{cases}.$$

Therefore,

$$\max_x u(x,t) = \begin{cases} 0 & t \leq 0 \\ t & 0 < t < \frac{a}{c} \\ \frac{a}{c} & t \geq \frac{a}{c} \end{cases}.$$