Exercise 7

If both ϕ and ψ are odd functions of x, show that the solution $u(x,t)$ of the wave equation is also odd in x for all t.

Solution

The general solution to the initial value problem,

$$u_{tt} = c^2 u_{xx},\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

If ϕ and ψ are odd functions, then

$$\begin{aligned} \phi(-x) &= -\phi(x) \\ \psi(-x) &= -\psi(x). \end{aligned}$$

What we want to show is that $u(x,t)$ is odd as a result, that is, $u(-x,t) = -u(x,t)$.

$$u(-x,t) = \frac{1}{2}\left[\phi(-x+ct) + \phi(-x-ct)\right] + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)\,ds$$

Let $w = -s$ in the integral. Then $dw = -ds$.

$$u(-x,t) = \frac{1}{2}\left[-\phi(x-ct) - \phi(x+ct)\right] + \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-w)(-dw)$$

Switching the limits of integration adds another minus sign.

$$u(-x,t) = -\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] - \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(w)\,dw$$

$$u(-x,t) = -\left\{\frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(w)\,dw\right\}$$

$$u(-x,t) = -u(x,t)$$

Therefore, if both $\phi(x)$ and $\psi(x)$ are odd functions of x, then $u(x,t)$ is also odd for all x and t.