

Exercise 8

A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right) \quad (\text{“spherical wave equation”}).$$

- Change variables $v = ru$ to get the equation for v : $v_{tt} = c^2 v_{rr}$.
- Solve for v using (3) and thereby solve the spherical wave equation.
- Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r .

Solution

Part (a)

If we change variables to $v = ru$, we have to write expressions for the derivatives of u in terms of v .

$$\begin{aligned} v_t &= ru_t \\ v_{tt} &= ru_{tt} \quad \rightarrow \quad \frac{v_{tt}}{r} = u_{tt} \\ v_r &= u + ru_r \\ v_{rr} &= u_r + u_r + ru_{rr} = 2u_r + ru_{rr} \quad \rightarrow \quad \frac{v_{rr}}{r} = u_{rr} + \frac{2}{r}u_r \end{aligned}$$

As a result of making the change of variables, the PDE becomes

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right) \quad \rightarrow \quad \frac{v_{tt}}{r} = c^2 \left(\frac{v_{rr}}{r} \right).$$

Therefore,

$$v_{tt} = c^2 v_{rr}.$$

Part (b)

Equation (3) in the book tells us that the solution to the one-dimensional wave equation above is

$$v(r, t) = f(r - ct) + g(r + ct), \quad (3)$$

where f and g are arbitrary functions. Now that we know v , we can solve for u .

$$ru = f(r - ct) + g(r + ct)$$

Therefore, the solution to the spherical wave equation is

$$u(r, t) = \frac{f(r - ct) + g(r + ct)}{r}.$$

Part (c)

The unknown functions, f and g , are determined by using the initial conditions.

$$\begin{aligned} u(r, 0) = \phi(r) &\rightarrow \frac{1}{r}[f(r) + g(r)] = \phi(r) \\ u_t(r, 0) = \psi(r) &\rightarrow \frac{1}{r}[-cf'(r) + cg'(r)] = \psi(r) \end{aligned}$$

Multiply both sides of the first equation by cr and both sides of the second equation by r .

$$\begin{aligned} cf(r) + cg(r) &= cr\phi(r) \\ -cf'(r) + cg'(r) &= r\psi(r) \end{aligned}$$

Differentiate both sides of the first equation with respect to r .

$$\begin{aligned} cf'(r) + cg'(r) &= c[r\phi(r)]' & (1) \\ -cf'(r) + cg'(r) &= r\psi(r) & (2) \end{aligned}$$

Subtract both sides of equation (2) from those of equation (1).

$$2cf'(r) = c[r\phi(r)]' - r\psi(r).$$

Add both sides of equation (2) to those of equation (1).

$$2cg'(r) = c[r\phi(r)]' + r\psi(r).$$

Solve the previous two equations for f' and g' .

$$\begin{aligned} f'(r) &= \frac{1}{2}[r\phi(r)]' - \frac{1}{2c}r\psi(r) \\ g'(r) &= \frac{1}{2}[r\phi(r)]' + \frac{1}{2c}r\psi(r) \end{aligned}$$

Now integrate both sides of each equation to find f and g .

$$\begin{aligned} f(r) &= \frac{1}{2}[r\phi(r)] - \frac{1}{2c} \int^r s\psi(s) ds \\ g(r) &= \frac{1}{2}[r\phi(r)] + \frac{1}{2c} \int^r s\psi(s) ds \end{aligned}$$

What we solved for are actually $f(w)$ and $g(w)$, where w is any expression. Thus,

$$\begin{aligned} f(r - ct) &= \frac{1}{2}[(r - ct)\phi(r - ct)] - \frac{1}{2c} \int^{r-ct} s\psi(s) ds \\ g(r + ct) &= \frac{1}{2}[(r + ct)\phi(r + ct)] + \frac{1}{2c} \int^{r+ct} s\psi(s) ds. \end{aligned}$$

Plugging these formulas into $u(r, t)$, we obtain the solution to the initial value problem.

$$\begin{aligned} u(r, t) &= \frac{1}{r} [f(r - ct) + g(r + ct)] \\ &= \frac{1}{r} \left\{ \frac{1}{2} [(r - ct)\phi(r - ct)] - \frac{1}{2c} \int^{r-ct} s\psi(s) ds + \frac{1}{2} [(r + ct)\phi(r + ct)] + \frac{1}{2c} \int^{r+ct} s\psi(s) ds \right\} \\ &= \frac{1}{2r} [(r - ct)\phi(r - ct) + (r + ct)\phi(r + ct)] + \frac{1}{2cr} \left[\int^{r+ct} s\psi(s) ds - \int^{r-ct} s\psi(s) ds \right] \\ &= \frac{1}{2r} [(r - ct)\phi(r - ct) + (r + ct)\phi(r + ct)] + \frac{1}{2cr} \left[\int^{r+ct} s\psi(s) ds + \int_{r-ct} s\psi(s) ds \right] \end{aligned}$$

Therefore,

$$u(r, t) = \frac{1}{2r} [(r - ct)\phi(r - ct) + (r + ct)\phi(r + ct)] + \frac{1}{2cr} \int_{r-ct}^{r+ct} s\psi(s) ds.$$