Exercise 9

Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,\ u(x,0) = x^2,\ u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

Solution

Solution by Operator Factorization

By factoring the PDE and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by writing the left side as an operator acting on u, $\mathscr{L}u$.

$$(\partial_x^2 - 3\partial_x\partial_t - 4\partial_t^2)u = 0$$

Now factor the operator.

$$(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$$

If we let $v = (\partial_x + \partial_t)u$, then what remains is $(\partial_x - 4\partial_t)v = 0$. That is,

$$\begin{cases} u_x + u_t = v \\ v_x - 4v_t = 0 \end{cases}.$$

On the paths defined by

$$\frac{dt}{dx} = -4, \quad t(0,\xi) = \xi \tag{1}$$

the PDE on the bottom reduces to an ODE,

$$\frac{dv}{dx} = 0. \tag{2}$$

That is, $v = v(x,t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$v(x,\xi) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by integration gives

$$t = -4x + \xi.$$

Solve now for ξ.

$$\xi = t + 4x$$

Therefore,

$$v(x,t) = f(t + 4x).$$

We can check that this is the solution of the bottom PDE.

$$v_x = 4f'$$
$$v_t = f'$$

$v_x - 4v_t = 0$, so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

$$u_x + u_t = f(t + 4x).$$

On the paths defined by

$$\frac{dt}{dx} = 1, \quad t(0, \eta) = \eta \tag{3}$$

the PDE on top reduces to an ODE,

$$\frac{du}{dx} = f(t + 4x). \tag{4}$$

That is, $u = u(x,t)$ is constant on the characteristics defined by (3). Solving (3) by integration, we find that

$$t = x + \eta.$$

Solve now for η.

$$\eta = t - x$$

Now that we know t in terms of η, we can solve (4) for u.

$$\frac{du}{dx} = f(5x + \eta)$$

Integrate both sides now with respect to x.

$$u(x, \eta) = \int^x f(5s + \eta)\, ds + g(\eta),$$

where g is an arbitrary function of η. Now that the integration is done, change back to the original variables.

$$u(x,t) = \int^x f(5s + t - x)\, ds + g(t - x)$$

To solve the integral, use a substitution. Plug in $s = x$ to get the upper limit of integration.

$$w = 5s + t - x$$
$$dw = 5\, ds \quad \rightarrow \quad \frac{1}{5}\, dw = ds$$

It becomes

$$u(x,t) = \int^{t+4x} f(w) \frac{1}{5}\, dw + g(t - x)$$
$$u(x,t) = \frac{1}{5} F(t + 4x) + g(t - x).$$

Now that we have the general solution for $u(x,t)$, we use the initial conditions to determine the unknown functions, F and g.

$$u(x,0) = x^2 \quad \rightarrow \quad \frac{1}{5} F(4x) + g(-x) = x^2$$
$$u_t(x,0) = e^x \quad \rightarrow \quad \frac{1}{5} F'(4x) + g'(-x) = e^x$$

Differentiate both sides of the top equation with respect to x.

$$\frac{4}{5}F'(4x) - g'(-x) = 2x$$
$$\frac{1}{5}F'(4x) + g'(-x) = e^x$$

Adding these two equations gives us

$$F'(4x) = 2x + e^x.$$

Subtracting the two equations gives us

$$\frac{3}{5}F'(4x) - 2g'(-x) = 2x - e^x.$$

Make the substitution, $w = 4x$, in the equation for F'. It becomes

$$F'(w) = 2\left(\frac{w}{4}\right) + e^{w/4}.$$

Now integrate both sides to determine F.

$$F(w) = \frac{w^2}{4} + 4e^{w/4}$$

All that's left is to determine g'. Plug in the expression for F'.

$$\frac{3}{5}(2x + e^x) - 2g'(-x) = 2x - e^x$$

Solving for $g'(-x)$ yields

$$g'(-x) = \frac{2}{5}(2e^x - x).$$

Make the change of variables, $s = -x$, and then integrate to get g.

$$g'(s) = \frac{2}{5}(2e^{-s} + s)$$
$$g(s) = \frac{2}{5}\left(-2e^{-s} + \frac{1}{2}s^2\right)$$

We can choose any expressions we wish for w and s. Thus,

$$F(t + 4x) = \frac{1}{4}(t + 4x)^2 + 4e^{\frac{t+4x}{4}}$$
$$g(t - x) = \frac{2}{5}\left[-2e^{-(t-x)} + \frac{1}{2}(t - x)^2\right].$$

Plugging these into $u(x, t)$, we obtain the answer to the initial value problem.

$$u(x, t) = \frac{1}{5}\left\{\frac{1}{4}(t + 4x)^2 + 4e^{\frac{t+4x}{4}}\right\} + \frac{2}{5}\left[-2e^{-(t-x)} + \frac{1}{2}(t - x)^2\right]$$
$$u(x, t) = \frac{4}{5}\left(e^{\frac{t+4x}{4}} - e^{x-t}\right) + \frac{1}{5}\left[\frac{1}{4}(t + 4x)^2 + (t - x)^2\right]$$

Therefore,

$$u(x, t) = \frac{4}{5}e^{x-t}(e^{5t/4} - 1) + x^2 + \frac{t^2}{4}.$$

Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = -4$, $B = -3$, $C = 1$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\ \frac{dx}{dt} &= \frac{1}{-8}\left(-3 \pm \sqrt{9+16}\right) \\ \frac{dx}{dt} &= -\frac{1}{8}(-3 \pm 5) \\ \frac{dx}{dt} &= -\frac{1}{4} \quad \text{or} \quad \frac{dx}{dt} = 1. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 25$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xt-plane.

$$x = -\frac{1}{4}t + C_1 \quad \text{or} \quad x = t + C_2.$$

Solving for the constants of integration,

$$\begin{aligned} C_1 &= x + \frac{1}{4}t = \phi(x,t) \\ C_2 &= x - t = \psi(x,t). \end{aligned}$$

Now we make the change of variables, $\xi = \phi(x,t) = x + (1/4)t$ and $\eta = \psi(x,t) = x - t$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule,

$$\begin{aligned} A^* &= A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2 \\ B^* &= 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x \\ C^* &= A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2 \\ D^* &= A\xi_{tt} + B\xi_{xt} + C\xi_{xx} + D\xi_t + E\xi_x \\ E^* &= A\eta_{tt} + B\eta_{xt} + C\eta_{xx} + D\eta_t + E\eta_x \\ F^* &= F \\ G^* &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 25/4$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$\frac{25}{4}u_{\xi\eta} = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = 0.$$

This is known as the first canonical form of the PDE. We can solve it by integrating both sides with respect to η and then integrating both sides again with respect to ξ.

$$u_\xi = f(\xi),$$

where f is an arbitrary function of ξ.

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

where F and G are arbitrary functions of ξ and η, respectively. Now change back to the original variables with the substitutions, $\xi = x + (1/4)t$ and $\eta = x - t$, to get the general solution of the PDE.

$$u(x,t) = F\left(x + \frac{1}{4}t\right) + G(x - t)$$

We determine these unknown functions by using the initial conditions.

$$\begin{aligned} u(x,0) = x^2 \quad &\rightarrow \quad F(x) + G(x) = x^2 \\ u_t(x,0) = e^x \quad &\rightarrow \quad \frac{1}{4}F'(x) - G'(x) = e^x \end{aligned}$$

Differentiate both sides of the top equation with respect to x.

$$\begin{aligned} F'(x) + G'(x) &= 2x \\ \frac{1}{4}F'(x) - G'(x) &= e^x \end{aligned}$$

Adding these two equations gives us

$$\frac{5}{4}F'(x) = 2x + e^x.$$

Subtracting the two equations gives us

$$\frac{3}{4}F'(x) + 2G'(x) = 2x - e^x.$$

To solve for F, multiply both sides of the equation by $4/5$ and then integrate both sides.

$$F'(x) = \frac{4}{5}(2x + e^x)$$

$$F(x) = \frac{4}{5}(x^2 + e^x)$$

Now solve for G by plugging in F', isolating G', and then integrating both sides.

$$\frac{3}{4} \cdot \frac{4}{5}(2x + e^x) + 2G'(x) = 2x - e^x$$

$$G'(x) = \frac{2}{5}(x - 2e^x)$$

$$G(x) = \frac{1}{5}(x^2 - 4e^x)$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression. Thus,

$$F\left(x+\frac{1}{4}t\right) = \frac{4}{5}\left[\left(x+\frac{1}{4}t\right)^2 + e^{x+\frac{1}{4}t}\right]$$
$$G(x-t) = \frac{1}{5}[(x-t)^2 - 4e^{x-t}].$$

Plugging these into $u(x,t)$, we obtain the answer to the initial value problem.

$$u(x,t) = \frac{4}{5}\left[\left(x+\frac{1}{4}t\right)^2 + e^{x+\frac{1}{4}t}\right] + \frac{1}{5}[(x-t)^2 - 4e^{x-t}]$$

$$u(x,t) = \frac{4}{5}\left(e^{x+\frac{1}{4}t} - e^{x-t}\right) + \frac{1}{5}\left[4\left(x+\frac{1}{4}t\right)^2 + (x-t)^2\right]$$

Therefore,

$$u(x,t) = \frac{4}{5}e^{x-t}(e^{5t/4}-1) + x^2 + \frac{t^2}{4}.$$

This is the same answer that we obtained with operator factorization. We can check that this is the solution to the PDE.

$$u_t = \frac{1}{5}e^{x-t}(e^{5t/4}+4) + \frac{t}{2}$$
$$u_{tt} = \frac{1}{20}e^{x-t}(e^{5t/4}-16) + \frac{1}{2}$$
$$u_x = \frac{4}{5}e^{x-t}(e^{5t/4}-1) + 2x$$
$$u_{xt} = \frac{1}{5}e^{x-t}(e^{5t/4}+4)$$
$$u_{xx} = \frac{4}{5}e^{x-t}(e^{5t/4}-1) + 2$$

$u_{xx} - 3u_{xt} - 4u_{tt} = 0$, so this is the solution to the PDE. Now check the initial conditions.

$$u(x,0) = \frac{4}{5}e^x(1-1) + x^2 + 0 = x^2$$
$$u_t(x,0) = \frac{1}{5}e^x(1+4) = e^x$$

To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath's "Linear Partial Differential Equations for Scientists and Engineers, 4th Edition."