Exercise 1

Solve $u_{tt} = c^2 u_{xx},\ u(x,0) = e^x,\ u_t(x,0) = \sin x$.

Solution

Solution by Operator Factorization

By factoring the wave equation and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by bringing all terms of the PDE to one side.

$$u_{tt} - c^2 u_{xx} = 0$$

Write the left side as an operator acting on u, $\mathscr{L}u$.

$$(\partial_t^2 - c^2 \partial_x^2)u = 0$$

Now factor the operator.

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

If we let $v = (\partial_t + c\partial_x)u$, then what remains is $(\partial_t - c\partial_x)v = 0$. That is,

$$\begin{cases} u_t + cu_x = v \\ v_t - cv_x = 0 \end{cases}.$$

On the paths defined by

$$\frac{dx}{dt} = -c, \quad x(\xi, 0) = \xi \tag{1}$$

the PDE on the bottom reduces to an ODE,

$$\frac{dv}{dt} = 0. \tag{2}$$

That is, $v = v(x,t)$ is constant on the characteristics defined by (1). Integrating (2), we find that

$$v(\xi, t) = f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Solving (1) by integration gives

$$x = -ct + \xi$$

Solve now for ξ.

$$\xi = x + ct$$

Therefore,

$$v(x,t) = f(x + ct).$$

We can check that this is the solution of the bottom PDE.

$$v_t = cf'$$
$$v_x = f'$$

$v_t - cv_x = 0$, so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

$$u_t + cu_x = f(x + ct).$$

On the paths defined by

$$\frac{dx}{dt} = c, \quad x(\eta, 0) = \eta \tag{3}$$

the PDE on top reduces to an ODE,

$$\frac{du}{dt} = f(x + ct). \tag{4}$$

That is, $u = u(x,t)$ is constant on the characteristics defined by (3). Solving (3) by integration, we find that

$$x = ct + \eta.$$

Solve now for η.

$$\eta = x - ct$$

Now that we know x in terms of η, we can solve (4) for u.

$$\frac{du}{dt} = f(2ct + \eta)$$

Integrate both sides now with respect to t.

$$u(\eta, t) = \int^t f(2cs + \eta)\, ds + g(\eta),$$

where g is an arbitrary function of η. Now that the integration is done, change back to the original variables.

$$u(x,t) = \int^t f(2cs + x - ct)\, ds + g(x - ct)$$

To solve the integral, use a substitution. Plug in $s = t$ to get the upper limit of integration.

$$\begin{gathered} w = 2cs + x - ct \\ dw = 2c\, ds \quad \rightarrow \quad \frac{1}{2c}\, dw = ds \end{gathered}$$

It becomes

$$\begin{gathered} u(x,t) = \int^{x+ct} f(w) \frac{1}{2c}\, dw + g(x - ct) \\ u(x,t) = \frac{1}{2c} F(x + ct) + g(x - ct). \end{gathered}$$

Now that we have the general solution for $u(x,t)$, we use the initial conditions to determine the unknown functions, F and g.

$$\begin{aligned} u(x,0) = e^x \quad &\rightarrow \quad \frac{1}{2c} F(x) + g(x) = e^x \\ u_t(x,0) = \sin x \quad &\rightarrow \quad \frac{1}{2} F'(x) - cg'(x) = \sin x \end{aligned}$$

Multiply both sides of the top equation by c and differentiate both sides with respect to x.

$$\frac{1}{2}F'(x) + cg'(x) = ce^x$$
$$\frac{1}{2}F'(x) - cg'(x) = \sin x$$

Adding these two equations gives us

$$F'(x) = ce^x + \sin x.$$

Subtracting the two equations gives us

$$2cg'(x) = ce^x - \sin x.$$

Divide both sides by $2c$ to isolate g'.

$$g'(x) = \frac{1}{2}e^x - \frac{1}{2c}\sin x$$

Solve for F and g by integrating both sides.

$$F(x) = ce^x - \cos x$$
$$g(x) = \frac{1}{2}e^x + \frac{1}{2c}\cos x$$

What we solved for are actually $F(w)$ and $g(w)$, where w is any expression. Thus,

$$F(x+ct) = ce^{x+ct} - \cos(x+ct)$$
$$g(x-ct) = \frac{1}{2}e^{x-ct} + \frac{1}{2c}\cos(x-ct).$$

Plugging these into $u(x,t)$, we obtain the answer to the initial value problem.

$$u(x,t) = \frac{1}{2c}[ce^{x+ct} - \cos(x+ct)] + \left[\frac{1}{2}e^{x-ct} + \frac{1}{2c}\cos(x-ct)\right]$$

$$u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct)\right]$$

$$u(x,t) = \frac{1}{2}\left(2e^x\cosh ct\right) + \frac{1}{2c}(2\sin x\sin ct)$$

Therefore,

$$u(x,t) = e^x\cosh ct + \frac{1}{c}\sin x\sin ct.$$

Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\ \frac{dx}{dt} &= \frac{1}{2}\left(\pm\sqrt{0 + 4c^2}\right) \\ \frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xt-plane.

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solving for the constants of integration,

$$\begin{aligned} C_1 &= x - ct = \phi(x, t) \\ C_2 &= x + ct = \psi(x, t). \end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, t) = x - ct$ and $\eta = \psi(x, t) = x + ct$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule,

$$\begin{aligned} A^* &= A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2 \\ B^* &= 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x \\ C^* &= A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2 \\ D^* &= A\xi_{tt} + B\xi_{xt} + C\xi_{xx} + D\xi_t + E\xi_x \\ E^* &= A\eta_{tt} + B\eta_{xt} + C\eta_{xx} + D\eta_t + E\eta_x \\ F^* &= F \\ G^* &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -4c^2$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-4c^2 u_{\xi\eta} = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = 0.$$

This is known as the first canonical form of the wave equation. We can solve it by integrating both sides with respect to η and then integrating both sides again with respect to ξ.

$$u_\xi = f(\xi),$$

where f is an arbitrary function of ξ.

$$u(\xi, \eta) = F(\xi) + G(\eta),$$

where F and G are arbitrary functions of ξ and η, respectively. Now change back to the original variables with the substitutions, $\xi = x - ct$ and $\eta = x + ct$, to get the general solution of the wave equation.

$$u(x, t) = F(x - ct) + G(x + ct)$$

We determine these unknown functions by using the initial conditions.

$$\begin{aligned} u(x, 0) = e^x &\quad \rightarrow \quad F(x) + G(x) = e^x \\ u_t(x, 0) = \sin x &\quad \rightarrow \quad -cF'(x) + cG'(x) = \sin x \end{aligned}$$

Multiply both sides of the top equation by c and differentiate both sides with respect to x.

$$\begin{aligned} cF'(x) + cG'(x) &= ce^x \\ -cF'(x) + cG'(x) &= \sin x \end{aligned}$$

Adding these two equations gives us

$$2cG'(x) = ce^x + \sin x.$$

Subtracting the two equations gives us

$$2cF'(x) = ce^x - \sin x.$$

Divide both sides by $2c$ to isolate F' and G'.

$$\begin{aligned} F'(x) &= \frac{1}{2}e^x - \frac{1}{2c}\sin x \\ G'(x) &= \frac{1}{2}e^x + \frac{1}{2c}\sin x \end{aligned}$$

Solve for F and G by integrating both sides.

$$\begin{aligned} F(x) &= \frac{1}{2}e^x + \frac{1}{2c}\cos x \\ G(x) &= \frac{1}{2}e^x - \frac{1}{2c}\cos x \end{aligned}$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression. Thus,

$$\begin{aligned} F(x + ct) &= \frac{1}{2}e^{x+ct} + \frac{1}{2c}\cos(x + ct) \\ G(x - ct) &= \frac{1}{2}e^{x-ct} - \frac{1}{2c}\cos(x - ct). \end{aligned}$$

Plugging these into $u(x, t)$, we obtain the answer to the initial value problem.

$$u(x, t) = \frac{1}{2}e^{x+ct} + \frac{1}{2c}\cos(x + ct) + \frac{1}{2}e^{x-ct} - \frac{1}{2c}\cos(x - ct)$$

$$u(x, t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\left[\cos(x - ct) - \cos(x + ct)\right]$$

$$u(x,t) = \frac{1}{2}\left(2e^x \cosh ct\right) + \frac{1}{2c}(2\sin x \sin ct)$$

Therefore,

$$u(x,t) = e^x \cosh ct + \frac{1}{c}\sin x \sin ct.$$

This is the same answer that we obtained with operator factorization. We can check that this is the solution to the wave equation.

$$\begin{aligned}
u_t &= ce^x \sinh ct + \sin x \cos ct \\
u_{tt} &= c^2 e^x \cosh ct - c \sin x \sin ct \\
u_x &= e^x \cosh ct + \frac{1}{c}\cos x \sin ct \\
u_{xx} &= e^x \cosh ct - \frac{1}{c}\sin x \sin ct
\end{aligned}$$

$u_{tt} = c^2 u_{xx}$, so this is the solution to the wave equation. Now check the initial conditions.

$$\begin{aligned}
u(x,0) &= e^x \cdot 1 + 0 = e^x \\
u_t(x,0) &= ce^x \cdot 0 + \sin x \cdot 1 = \sin x
\end{aligned}$$

To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath's "Linear Partial Differential Equations for Scientists and Engineers, 4th Edition."