Exercise 11

Find the general solution of \( 3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t) \).

Solution by Operator Factorization

By factoring the PDE and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by writing the left side as an operator acting on \( u \), \( \mathcal{L}u \).

\[
(\partial_x^2 + \partial_x \partial_t - 20\partial_t^2)u = \sin(x + t)
\]

Now factor the operator.

\[
(3\partial_t + \partial_x)(\partial_t + 3\partial_x)u = \sin(x + t)
\]

If we let \( v = (\partial_t + 3\partial_x)u \), then what remains is \( (3\partial_t + \partial_x)v = \sin(x + t) \). Divide both sides of this equation by 3. That is,

\[
\begin{cases}
  u_t + 3u_x = v \\
  v_t + \frac{1}{3}v_x = \frac{1}{3} \sin(x + t)
\end{cases}
\]

On the paths defined by

\[
\frac{dx}{dt} = \frac{1}{3}; \quad x(\xi, 0) = \xi
\]

the PDE on the bottom reduces to an ODE,

\[
\frac{dv}{dt} = \frac{1}{3} \sin(x + t).
\]

That is, \( v = v(x, t) \) is constant on the characteristics defined by (1). Solving (1) by integration gives

\[ x = \frac{1}{3}t + \xi. \]

Solve now for \( \xi \).

\[ \xi = x - \frac{1}{3}t \]

Now that we can write \( x \) in terms of \( t \), we can solve (2) for \( v \).

\[ \frac{dv}{dt} = \frac{1}{3} \sin\left(\xi + \frac{4}{3}t\right) \]

Integrating this equation, we find that

\[ v(\xi, t) = \frac{1}{3} \int_t^0 \sin\left(\xi + \frac{4}{3}s\right) ds + f(\xi), \]

where \( f \) is an arbitrary function of the characteristic coordinate, \( \xi \). Therefore,

\[ v(x, t) = -\frac{1}{4} \cos(x + t) + f\left(x - \frac{1}{3}t\right). \]
We can check that this is the solution of the bottom PDE.

\[ v_x = \frac{1}{4} \sin(x + t) + f' \]

\[ v_t = \frac{1}{4} \sin(x + t) - \frac{1}{3} f' \]

3\(v_t + v_x = \sin(x + t)\), so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

\[ u_t + 3u_x = -\frac{1}{4} \cos(x + t) + f \left( x - \frac{1}{3} t \right). \]

On the paths defined by

\[ \frac{dx}{dt} = 3, \quad x(\eta, 0) = \eta \] (3)

this PDE reduces to an ODE,

\[ \frac{du}{dt} = -\frac{1}{4} \cos(x + t) + f \left( x - \frac{1}{3} t \right). \] (4)

That is, \( u = u(x, t) \) is constant on the characteristics defined by (3). Solving (3) by integration, we find that

\[ x = 3t + \eta. \]

Solve now for \( \eta \).

\[ \eta = x - 3t \]

Now that we know \( x \) in terms of \( \eta \), we can solve (4) for \( u \).

\[ \frac{du}{dt} = -\frac{1}{4} \cos(\eta + 4t) + f \left( \eta + \frac{8}{3} t \right) \]

Integrate both sides now with respect to \( t \).

\[ u(\eta, t) = -\frac{1}{4} \int^t \cos(\eta + 4s) \, ds + \int^t f \left( \eta + \frac{8}{3} s \right) \, ds + g(\eta), \]

where \( g \) is an arbitrary function of \( \eta \).

\[ u(\eta, t) = -\frac{1}{16} \sin(\eta + 4t) + \int^t f \left( \eta + \frac{8}{3} s \right) \, ds + g(\eta) \]

To solve the integral, use a substitution. Plug in \( s = t \) to get the upper limit of integration.

\[ w = \eta + \frac{8}{3} s \]

\[ dw = \frac{8}{3} \, ds \quad \rightarrow \quad \frac{3}{8} \, dw = ds \]

Doing this results in the following.

\[ u(\eta, t) = -\frac{1}{16} \sin(\eta + 4t) + \int^{\eta + \frac{4}{3} t} f(w) \frac{3}{8} \, dw + g(\eta) \]
\[ u(\eta, t) = -\frac{1}{16} \sin(\eta + 4t) + \frac{3}{8} \int_{\eta}^{\eta + \frac{8}{3}t} f(w) \, dw + g(\eta) \]

\[ u(\eta, t) = -\frac{1}{16} \sin(\eta + 4t) + \frac{3}{8} F \left( \eta + \frac{8}{3}t \right) + g(\eta) \]

Now that the integration is done, change back to the original variables. This is the general solution to the PDE.

\[ u(x, t) = -\frac{1}{16} \sin(x + t) + \frac{3}{8} F \left( x - \frac{1}{3}t \right) + g(x - 3t) \]

We can check that it is so.

\[ u_t = -\frac{1}{16} \cos(x + t) - \frac{1}{8} F'(x - \frac{1}{3}t) - 3g'(x - 3t) \]
\[ u_{tt} = \frac{1}{16} \sin(x + t) + \frac{1}{24} F''(x - \frac{1}{3}t) + 9g''(x - 3t) \]
\[ u_x = -\frac{1}{16} \cos(x + t) + \frac{3}{8} F'(x - \frac{1}{3}t) + g'(x - 3t) \]
\[ u_{xx} = \frac{1}{16} \sin(x + t) + \frac{3}{8} F''(x - \frac{1}{3}t) + g''(x - 3t) \]
\[ u_{xt} = \frac{1}{16} \sin(x + t) - \frac{1}{8} F''(x - \frac{1}{3}t) - 3g''(x - 3t) \]

Hence, \( 3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t) \), which means this is the correct solution to the PDE.
Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE, \( Au_{tt} + Bu_{xt} + Cu_{xx} + Du_{t} + Eu_{x} + Fu = G \), we see that \( A = 3, B = 10, C = 3, D = 0, E = 0, F = 0, \) and \( G = \sin(x + t) \). The characteristic equations of this PDE are given by

\[
\frac{dx}{dt} = \frac{1}{2A} \left( B \pm \sqrt{B^2 - 4AC} \right)
\]
\[
\frac{dx}{dt} = \frac{1}{6} (10 \pm \sqrt{100 - 36})
\]
\[
\frac{dx}{dt} = \frac{1}{6} (10 \pm 8)
\]
\[
\frac{dx}{dt} = 3 \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{3}.
\]

Note that the discriminant, \( B^2 - 4AC = 64 \), is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the \( xt \)-plane.

\[
x = 3t + C_1 \quad \text{or} \quad x = \frac{1}{3} t + C_2.
\]

Solving for the constants of integration,

\[
C_1 = x - 3t = \phi(x, t)
\]
\[
C_2 = x - \frac{1}{3} t = \psi(x, t).
\]

Now we make the change of variables, \( \xi = \phi(x, t) = x - 3t \) and \( \eta = \psi(x, t) = x - (1/3)t \), so that the PDE takes the simplest form. The old variables, \( x \) and \( t \), in terms of the new ones are

\[
t = \frac{3}{8}(\eta - \xi)
\]
\[
x = \frac{1}{8}(9\eta - \xi).
\]

With these new variables the PDE becomes

\[
A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_{\xi} + E^* u_{\eta} + F^* u = G^*,
\]

where, using the chain rule,

\[
A^* = A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2
\]
\[
B^* = 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x
\]
\[
C^* = A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2
\]
\[
D^* = A\xi_t + B\xi_xt + C\xi_x + D\xi_t + E\xi_x
\]
\[
E^* = A\eta_t + B\eta_xt + C\eta_x + D\eta_t + E\eta_x
\]
\[
F^* = F
\]
\[
G^* = G.
\]
Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -64/3$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = \sin(x + t) = \sin[(1/2)(3\eta - \xi)]$. Thus, the PDE simplifies to

$$-\frac{64}{3} u_{\xi\eta} = \sin\left(\frac{1}{2}(3\eta - \xi)\right).$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = -\frac{3}{64} \sin\left(\frac{1}{2}(3\eta - \xi)\right).$$

This is known as the first canonical form of the PDE. We can solve it by integrating both sides with respect to $\eta$ and then integrating both sides again with respect to $\xi$.

$$u_{\xi} = -\frac{3}{64} \int_{\eta}^{\xi} \sin\left(\frac{1}{2}(3\eta - \xi)\right) ds + f(\xi),$$

where $f$ is an arbitrary function of $\xi$. Use a substitution to solve the integral. Plug in $s = \eta$ to get the new upper limit of integration.

$$w = \frac{1}{2}(3s - \xi)$$

$$dw = \frac{3}{2} ds \rightarrow \frac{2}{3} dw = ds$$

Doing this results in the following.

$$u_{\xi} = -\frac{3}{64} \int_{\eta}^{\xi} \sin w \left(\frac{2}{3}\right) dw + f(\xi)$$

$$u_{\xi} = \frac{1}{32} \cos\left(\frac{1}{2}(3\eta - \xi)\right) + f(\xi)$$

Now we integrate with respect to $\xi$.

$$u(\xi, \eta) = \frac{1}{32} \int_{\xi}^{\xi} \cos\left(\frac{1}{2}(3\eta - s)\right) ds + \int_{\xi}^{\xi} f(s) ds + g(\eta),$$

where $g$ is an arbitrary function of $\eta$. Use a substitution to solve the integral. Plug in $s = \xi$ to get the new upper limit of integration.

$$w = \frac{1}{2}(3\eta - s)$$

$$dw = -\frac{1}{2} ds \rightarrow -2 dw = ds$$

It becomes

$$u(\xi, \eta) = \frac{1}{32} \int_{\xi}^{\xi} \cos w(-2) dw + F(\xi) + g(\eta),$$

where $F$ is an arbitrary function of $\xi$.

$$u(\xi, \eta) = -\frac{1}{16} \sin\left(\frac{1}{2}(3\eta - \xi)\right) + F(\xi) + g(\eta)$$

Now change back to the original variables with the substitutions, $\xi = x - 3t$ and $\eta = x - (1/3)t$, to get the general solution of the PDE.

$$u(x, t) = -\frac{1}{16} \sin(x + t) + F(x - 3t) + g\left(x - \frac{1}{3}t\right)$$

www.stemjock.com
We can check that this is the general solution to the PDE.

\[ u_t = -\frac{1}{16} \cos(x + t) - 3F'(x - 3t) - \frac{1}{3} g' \left( x - \frac{1}{3} t \right) \]

\[ u_{tt} = \frac{1}{16} \sin(x + t) + 9F''(x - 3t) + \frac{1}{9} g'' \left( x - \frac{1}{3} t \right) \]

\[ u_x = -\frac{1}{16} \cos(x + t) + F'(x - 3t) + g' \left( x - \frac{1}{3} t \right) \]

\[ u_{xx} = \frac{1}{16} \sin(x + t) + F''(x - 3t) + g'' \left( x - \frac{1}{3} t \right) \]

\[ u_{xt} = \frac{1}{16} \sin(x + t) - 3F''(x - 3t) - \frac{1}{3} g'' \left( x - \frac{1}{3} t \right) \]

Hence, \( 3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t), \) which means this is the correct solution. To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath’s “Linear Partial Differential Equations for Scientists and Engineers, 4th Edition.”