Exercise 11

Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

Solution

Solution by Operator Factorization

By factoring the PDE and making a substitution, we can write it as an equivalent system of uncoupled first-order PDEs and solve it using the methods of the previous chapter. Start off by writing the left side as an operator acting on u, $\mathscr{L}u$.

$$(\partial_x^2 + \partial_x\partial_t - 20\partial_t^2)u = \sin(x+t)$$

Now factor the operator.

$$(3\partial_t + \partial_x)(\partial_t + 3\partial_x)u = \sin(x+t)$$

If we let $v = (\partial_t + 3\partial_x)u$, then what remains is $(3\partial_t + \partial_x)v = \sin(x+t)$. Divide both sides of this equation by 3. That is,

$$\begin{cases} u_t + 3u_x = v \\ v_t + \frac{1}{3}v_x = \frac{1}{3}\sin(x+t) \end{cases}.$$

On the paths defined by

$$\frac{dx}{dt} = \frac{1}{3}, \quad x(\xi, 0) = \xi \tag{1}$$

the PDE on the bottom reduces to an ODE,

$$\frac{dv}{dt} = \frac{1}{3}\sin(x+t). \tag{2}$$

That is, $v = v(x,t)$ is constant on the characteristics defined by (1). Solving (1) by integration gives

$$x = \frac{1}{3}t + \xi.$$

Solve now for ξ.

$$\xi = x - \frac{1}{3}t$$

Now that we can write x in terms of t, we can solve (2) for v.

$$\frac{dv}{dt} = \frac{1}{3}\sin\left(\xi + \frac{4}{3}t\right)$$

Integrating this equation, we find that

$$v(\xi, t) = \frac{1}{3}\int^t \sin\left(\xi + \frac{4}{3}s\right)\, ds + f(\xi),$$

where f is an arbitrary function of the characteristic coordinate, ξ. Therefore,

$$v(x,t) = -\frac{1}{4}\cos(x+t) + f\left(x - \frac{1}{3}t\right).$$

We can check that this is the solution of the bottom PDE.

$$v_x = \frac{1}{4}\sin(x+t) + f'$$
$$v_t = \frac{1}{4}\sin(x+t) - \frac{1}{3}f'$$

$3v_t + v_x = \sin(x+t)$, so this is the correct solution. Now we use this solution to solve the PDE on the top. Our task is to solve

$$u_t + 3u_x = -\frac{1}{4}\cos(x+t) + f\left(x - \frac{1}{3}t\right).$$

On the paths defined by

$$\frac{dx}{dt} = 3, \quad x(\eta, 0) = \eta \tag{3}$$

this PDE reduces to an ODE,

$$\frac{du}{dt} = -\frac{1}{4}\cos(x+t) + f\left(x - \frac{1}{3}t\right). \tag{4}$$

That is, $u = u(x,t)$ is constant on the characteristics defined by (3). Solving (3) by integration, we find that

$$x = 3t + \eta.$$

Solve now for η.

$$\eta = x - 3t$$

Now that we know x in terms of η, we can solve (4) for u.

$$\frac{du}{dt} = -\frac{1}{4}\cos(\eta + 4t) + f\left(\eta + \frac{8}{3}t\right)$$

Integrate both sides now with respect to t.

$$u(\eta, t) = -\frac{1}{4}\int^t \cos(\eta + 4s)\,ds + \int^t f\left(\eta + \frac{8}{3}s\right)\,ds + g(\eta),$$

where g is an arbitrary function of η.

$$u(\eta, t) = -\frac{1}{16}\sin(\eta + 4t) + \int^t f\left(\eta + \frac{8}{3}s\right)\,ds + g(\eta)$$

To solve the integral, use a substitution. Plug in $s = t$ to get the upper limit of integration.

$$w = \eta + \frac{8}{3}s$$
$$dw = \frac{8}{3}\,ds \quad \to \quad \frac{3}{8}\,dw = ds$$

Doing this results in the following.

$$u(\eta, t) = -\frac{1}{16}\sin(\eta + 4t) + \int^{\eta + \frac{8}{3}t} f(w)\frac{3}{8}\,dw + g(\eta)$$

$$u(\eta, t) = -\frac{1}{16}\sin(\eta + 4t) + \frac{3}{8}\int^{\eta + \frac{8}{3}t} f(w)\,dw + g(\eta)$$

$$u(\eta, t) = -\frac{1}{16}\sin(\eta + 4t) + \frac{3}{8}F\left(\eta + \frac{8}{3}t\right) + g(\eta)$$

Now that the integration is done, change back to the original variables. This is the general solution to the PDE.

$$u(x, t) = -\frac{1}{16}\sin(x + t) + \frac{3}{8}F\left(x - \frac{1}{3}t\right) + g(x - 3t)$$

We can check that it is so.

$$u_t = -\frac{1}{16}\cos(x + t) - \frac{1}{8}F'\left(x - \frac{1}{3}t\right) - 3g'(x - 3t)$$

$$u_{tt} = \frac{1}{16}\sin(x + t) + \frac{1}{24}F''\left(x - \frac{1}{3}t\right) + 9g''(x - 3t)$$

$$u_x = -\frac{1}{16}\cos(x + t) + \frac{3}{8}F'\left(x - \frac{1}{3}t\right) + g'(x - 3t)$$

$$u_{xx} = \frac{1}{16}\sin(x + t) + \frac{3}{8}F''\left(x - \frac{1}{3}t\right) + g''(x - 3t)$$

$$u_{xt} = \frac{1}{16}\sin(x + t) - \frac{1}{8}F''\left(x - \frac{1}{3}t\right) - 3g''(x - 3t)$$

Hence, $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$, which means this is the correct solution to the PDE.

Solution by the Method of Characteristics

Comparing this equation with the general form of a second-order PDE, $Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 3$, $B = 10$, $C = 3$, $D = 0$, $E = 0$, $F = 0$, and $G = \sin(x + t)$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dx}{dt} &= \frac{1}{6}\left(10 \pm \sqrt{100 - 36}\right) \\
\frac{dx}{dt} &= \frac{1}{6}(10 \pm 8) \\
\frac{dx}{dt} &= 3 \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{3}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 64$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xt-plane.

$$x = 3t + C_1 \quad \text{or} \quad x = \frac{1}{3}t + C_2.$$

Solving for the constants of integration,

$$\begin{aligned}
C_1 &= x - 3t = \phi(x, t) \\
C_2 &= x - \frac{1}{3}t = \psi(x, t).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, t) = x - 3t$ and $\eta = \psi(x, t) = x - (1/3)t$, so that the PDE takes the simplest form. The old variables, x and t, in terms of the new ones are

$$\begin{aligned}
t &= \frac{3}{8}(\eta - \xi) \\
x &= \frac{1}{8}(9\eta - \xi).
\end{aligned}$$

With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule,

$$\begin{aligned}
A^* &= A\xi_t^2 + B\xi_t\xi_x + C\xi_x^2 \\
B^* &= 2A\xi_t\eta_t + B(\xi_t\eta_x + \xi_x\eta_t) + 2C\xi_x\eta_x \\
C^* &= A\eta_t^2 + B\eta_t\eta_x + C\eta_x^2 \\
D^* &= A\xi_{tt} + B\xi_{xt} + C\xi_{xx} + D\xi_t + E\xi_x \\
E^* &= A\eta_{tt} + B\eta_{xt} + C\eta_{xx} + D\eta_t + E\eta_x \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -64/3$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = \sin(x+t) = \sin[(1/2)(3\eta - \xi)]$. Thus, the PDE simplifies to

$$-\frac{64}{3}u_{\xi\eta} = \sin\frac{1}{2}(3\eta - \xi).$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = -\frac{3}{64}\sin\frac{1}{2}(3\eta - \xi).$$

This is known as the first canonical form of the PDE. We can solve it by integrating both sides with respect to η and then integrating both sides again with respect to ξ.

$$u_\xi = -\frac{3}{64}\int^\eta \sin\frac{1}{2}(3\eta - \xi)\,ds + f(\xi),$$

where f is an arbitrary function of ξ. Use a substitution to solve the integral. Plug in $s = \eta$ to get the new upper limit of integration.

$$w = \frac{1}{2}(3s - \xi)$$

$$dw = \frac{3}{2}\,ds \quad \rightarrow \quad \frac{2}{3}\,dw = ds$$

Doing this results in the following.

$$u_\xi = -\frac{3}{64}\int^{\frac{1}{2}(3\eta-\xi)} \sin w\left(\frac{2}{3}\right)\,dw + f(\xi)$$

$$u_\xi = \frac{1}{32}\cos\frac{1}{2}(3\eta - \xi) + f(\xi)$$

Now we integrate with respect to ξ.

$$u(\xi,\eta) = \frac{1}{32}\int^\xi \cos\frac{1}{2}(3\eta - s)\,ds + \int^\xi f(s)\,ds + g(\eta),$$

where g is an arbitrary function of η. Use a substitution to solve the integral. Plug in $s = \xi$ to get the new upper limit of integration.

$$w = \frac{1}{2}(3\eta - s)$$

$$dw = -\frac{1}{2}\,ds \quad \rightarrow \quad -2\,dw = ds$$

It becomes

$$u(\xi,\eta) = \frac{1}{32}\int^{\frac{1}{2}(3\eta-\xi)} \cos w(-2)\,dw + F(\xi) + g(\eta),$$

where F is an arbitrary function of ξ.

$$u(\xi,\eta) = -\frac{1}{16}\sin\frac{1}{2}(3\eta - \xi) + F(\xi) + g(\eta)$$

Now change back to the original variables with the substitutions, $\xi = x - 3t$ and $\eta = x - (1/3)t$, to get the general solution of the PDE.

$$u(x,t) = -\frac{1}{16}\sin(x+t) + F(x - 3t) + g\left(x - \frac{1}{3}t\right)$$

We can check that this is the general solution to the PDE.

$$\begin{aligned}
u_t &= -\frac{1}{16}\cos(x+t) - 3F'(x-3t) - \frac{1}{3}g'\left(x - \frac{1}{3}t\right) \\
u_{tt} &= \frac{1}{16}\sin(x+t) + 9F''(x-3t) + \frac{1}{9}g''\left(x - \frac{1}{3}t\right) \\
u_x &= -\frac{1}{16}\cos(x+t) + F'(x-3t) + g'\left(x - \frac{1}{3}t\right) \\
u_{xx} &= \frac{1}{16}\sin(x+t) + F''(x-3t) + g''\left(x - \frac{1}{3}t\right) \\
u_{xt} &= \frac{1}{16}\sin(x+t) - 3F''(x-3t) - \frac{1}{3}g''\left(x - \frac{1}{3}t\right)
\end{aligned}$$

Hence, $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$, which means this is the correct solution. To learn about canonical forms and the method of characteristics for second-order PDEs, read chapter 4 of Debnath's "Linear Partial Differential Equations for Scientists and Engineers, 4th Edition."