Exercise 5

(The hammer blow) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile ($u$ versus $x$) at each of the successive instants $t = a/2c$, $a/c$, $3a/2c$, $2a/c$, and $5a/c$.

[Hint: Calculate

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds = \frac{1}{2c} \{\text{length of } (x - ct, x + ct) \cap (-a, a)\}.$$  

Then $u(x, a/2c) = (1/2c) \{\text{length of } (x - a/2, x + a/2) \cap (-a, a)\}$. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

Solution

The goal for this exercise is to plot the solution to the wave equation with initial conditions $u$ versus $x$ at five instants in time:

$$u_{tt} = c^2 u_{xx}, \quad u(x, 0) = 0, \quad u_t(x, 0) = \begin{cases} 1 & |x| < a \\ 0 & |x| \geq a \end{cases}.$$  

The solution to the wave equation with initial conditions, $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$, is given by D’Alembert’s formula,

$$u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds.$$  

Since $\phi(x) = 0$, the solution reduces to

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds.$$  

Following the hint, we will set $t$ equal to each of the five values and then determine

$$\frac{1}{2c} \{\text{length of } (x - ct, x + ct) \cap (-a, a)\}$$  

for various intervals of $x$ in order to evaluate $u(x, t)$. The reason we only care about the length from $x - ct$ to $x + ct$ here is because the integrand $\psi(x)$ is equal to 1 from $x = -a$ to $x = a$ and 0 everywhere else.

The String Profile for $t = \frac{a}{2c}$

Setting $t = \frac{a}{2c}$, we find that $x \pm ct = x \pm \frac{a}{2}$. The first interval of $x$ we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + \frac{a}{2}$, touches the left end of the interval, $(-a, a)$, at the highest value of $x$.

For $x < -\frac{3a}{2}$,  

$$\left( x - \frac{a}{2}, x + \frac{a}{2} \right) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset$$  

$$\rightarrow \quad u \left( x, \frac{a}{2c} \right) = \frac{1}{2c} \int_{-a}^{-a} 1 \, ds = 0$$

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The next interval of $x$ we consider is the one where the right-most point, $x + \frac{a}{2}$, is inside the interval, $(-a, a)$, but the left-most point, $x - \frac{a}{2}$, is not. The distance from $x - \frac{a}{2}$ to $x + \frac{a}{2}$ is $a$, so the next interval of $x$ will be from where we left off, $-\frac{3a}{2}$, to $-\frac{a}{2}$, where the left-most point touches the left end of the interval.

For $-\frac{3a}{2} < x < -\frac{a}{2}$, \( (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a, a) = \left(-a, x + \frac{a}{2}\right) \)

\[ \Rightarrow u \left(x, \frac{a}{2c}\right) = \frac{1}{2c} \int_{-a}^{x + \frac{a}{2}} 1 \, ds = \frac{1}{2c} \left(x + \frac{3a}{2}\right) \]

The next interval of $x$ we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are inside the interval, $(-a, a)$. This interval of $x$ will start from where we left off, $-\frac{a}{2}$, and go to $\frac{a}{2}$. The left-most point, $x - \frac{a}{2}$, touches the left end of the interval, $(-a, a)$, at the lowest value of $x$, and the right-most point, $x + \frac{a}{2}$, touches the right end of the interval, $(-a, a)$, at the highest value of $x$.

For $-\frac{a}{2} < x < \frac{a}{2}$, \( (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a, a) = (x - \frac{a}{2}, x + \frac{a}{2}) \)

\[ \Rightarrow u \left(x, \frac{a}{2c}\right) = \frac{1}{2c} \int_{x - \frac{a}{2}}^{x + \frac{a}{2}} 1 \, ds = \frac{a}{2c} \]

The next interval of $x$ we consider is the one where the right-most point, $x + \frac{a}{2}$, is outside the right end of the interval, $(-a, a)$, and the left-most point, $x - \frac{a}{2}$, is still inside of it. This interval of $x$ will start from where we left off, $\frac{a}{2}$, and go to $\frac{3a}{2}$, where the left-most point touches the right end of the interval, $(-a, a)$.

For $\frac{a}{2} < x < \frac{3a}{2}$, \( (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a, a) = \left(x - \frac{a}{2}, a\right) \)

\[ \Rightarrow u \left(x, \frac{a}{2c}\right) = \frac{1}{2c} \int_{x - \frac{a}{2}}^{a} 1 \, ds = \frac{1}{2c} \left(3a - x\right) \]

The last interval of $x$ we consider is the one where the left and right-most points, $x \pm \frac{a}{2}$, are outside the interval, $(-a, a)$, on the right side. The interval of $x$ will start from where we left off, $\frac{3a}{2}$, and go to $\infty$. The left-most point, $x - \frac{a}{2}$, touches the right end of the interval, $(-a, a)$, at the lowest value of $x$.

For $x > \frac{3a}{2}$, \( (x - \frac{a}{2}, x + \frac{a}{2}) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset \)

\[ \Rightarrow u \left(x, \frac{a}{2c}\right) = \int_{a}^{\infty} 1 \, ds = 0 \]

Now that we know $u \left(x, \frac{a}{2c}\right)$ for all $x$, it can be plotted.
The String Profile for $t = \frac{a}{2c}$

Setting $t = \frac{a}{2c}$, we find that $x \pm ct = x \pm a$. The first interval of $x$ we consider is the one where the left and right-most points, $x \pm a$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + a$, touches the left end of the interval, $(-a, a)$, at the highest value of $x$.

For $x < -2a$, 
$$(x - a, x + a) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset$$

$$\Rightarrow u \left( x, \frac{a}{c} \right) = \frac{1}{2c} \int_{-a}^{-a} 1 \, ds = 0$$

The next interval of $x$ we consider is the one where the right-most point, $x + a$, is inside the interval, $(-a, a)$, but the left-most point, $x - a$, is not. The distance from $x - a$ to $x + a$ is $2a$, so the next interval of $x$ will be from where we left off, $-2a$, to $0$.

For $-2a < x < 0$, 
$$(x - a, x + a) \cap (-a, a) = (-a, x + a)$$

$$\Rightarrow u \left( x, \frac{a}{c} \right) = \frac{1}{2c} \int_{-a}^{x+a} 1 \, ds = \frac{1}{2c} (x + 2a)$$

Since the interval, $(-a, a)$, is $2a$ units long as well, the next interval of $x$ we consider is the one where the right-most point, $x + a$, is outside the interval, $(-a, a)$, and the left-most point, $x - a$, is still inside it. This interval of $x$ will start from where we left off, $0$, and go to $2a$, where the left-most point touches the right end of the interval, $(-a, a)$.

For $0 < x < 2a$, 
$$(x - a, x + a) \cap (-a, a) = (x - a, a)$$

$$\Rightarrow u \left( x, \frac{a}{c} \right) = \frac{1}{2c} \int_{x-a}^{a} 1 \, ds = \frac{1}{2c} (2a - x)$$
The last interval of $x$ we consider is the one where the left and right-most points, $x \pm a$, are outside the interval $(-a, a)$ on the right side. The interval of $x$ will start from where we left off, $2a$, and go to $\infty$. The left-most point, $x - a$, touches the right end of the interval, $(-a, a)$, at the lowest value of $x$.

For $x > 2a$, \[ (x - a, x + a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset \]

\[ \rightarrow u \left( x, \frac{a}{c} \right) = \int_{a}^{\infty} 1 \, ds = 0 \]

Now that we know $u \left( x, \frac{a}{c} \right)$ for all $x$, it can be plotted.

![Figure 2: Plot of $u(x, t)$ when $t = \frac{a}{c}$.](image)

**The String Profile for $t = \frac{3a}{2c}$**

Setting $t = \frac{3a}{2c}$, we find that $x \pm ct = x \pm \frac{3a}{2}$. The first interval of $x$ we consider is the one where the left and right-most points, $x \pm \frac{3a}{2}$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + \frac{3a}{2}$, touches the left end of the interval $(-a, a)$ at the highest value of $x$.

For $x < -\frac{5a}{2}$, \[ \left( x - \frac{3a}{2}, x + \frac{3a}{2} \right) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset \]

\[ \rightarrow u \left( x, \frac{3a}{2c} \right) = \frac{1}{2c} \int_{-a}^{-\infty} 1 \, ds = 0 \]
The next interval of \( x \) we consider is the one where the right-most point, \( x + \frac{3a}{2} \), is inside the interval, \((-a, a)\), but the left-most point, \( x - \frac{3a}{2} \), is not. The distance from \( x - \frac{3a}{2} \) to \( x + \frac{3a}{2} \) is \( 3a \), which is bigger than the interval, \((-a,a)\). This means the next interval of \( x \) will be from where we left off, \(-\frac{5a}{2}\), to \(-\frac{a}{2}\), where the right-most point, \( x + \frac{3a}{2} \), touches the right end of the interval, \((-a,a)\).

For \(-\frac{5a}{2} < x < -\frac{a}{2}\), \((x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a, a) = (-a, x + \frac{3a}{2}) \)

\[ \rightarrow u \left( x, \frac{3a}{2c} \right) = \frac{1}{2c} \int_{-\frac{a}{2}}^{x + \frac{3a}{2}} ds = \frac{1}{2c} \left( x + \frac{5a}{2} \right) \]

Since the interval, \((-a,a)\), is only \( 2a \) units long, the next interval of \( x \) we consider is the one where the right-most point, \( x + \frac{3a}{2} \), is outside the interval, \((-a,a)\), on the right, and the left-most point, \( x - \frac{3a}{2} \), is still outside the left end. This interval of \( x \) will start from where we left off, \(-\frac{a}{2}\), and go to \( \frac{a}{2}\), where the left-most point touches the left end of the interval, \((-a,a)\).

For \(-\frac{a}{2} < x < \frac{a}{2}\), \((x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a, a) = (-a, a) \)

\[ \rightarrow u \left( x, \frac{3a}{2c} \right) = \frac{1}{2c} \int_{-a}^{\frac{a}{2}} ds = \frac{a}{c} \]

The next interval we consider is the one where the left-most point, \( x - \frac{3a}{2} \), is inside the left end of the interval, \((-a,a)\), and the right-most point, \( x + \frac{3a}{2} \), is outside \((-a,a)\) on the right side. The interval of \( x \) will start from where we left off, \( \frac{a}{2}\), and go to \( \frac{5a}{2}\), where the left-most point touches the right end of \((-a,a)\).

For \(\frac{a}{2} < x < \frac{5a}{2}\), \((x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a, a) = \left( x - \frac{3a}{2}, a \right) \)

\[ \rightarrow u \left( x, \frac{3a}{2c} \right) = \frac{1}{2c} \int_{x - \frac{3a}{2}}^{a} ds = \frac{1}{2c} \left( \frac{5a}{2} - x \right) \]

The last interval of \( x \) we consider is the one where the left and right-most points, \( x \pm \frac{3a}{2} \), are outside the interval, \((-a,a)\), on the right side. The interval of \( x \) will start from where we left off, \( \frac{5a}{2}\), and go to \( \infty \). The left-most point, \( x - \frac{3a}{2} \), touches the right end of the interval, \((-a,a)\), at the lowest value of \( x \).

For \( x > \frac{5a}{2}\), \((x - \frac{3a}{2}, x + \frac{3a}{2}) \cap (-a, a) = (a, \infty) \cap (-a,a) = \emptyset \)

\[ \rightarrow u \left( x, \frac{3a}{2c} \right) = \frac{1}{2c} \int_{a}^{\infty} ds = 0 \]

Now that we know \( u \left( x, \frac{3a}{2c} \right) \) for all \( x \), we can plot it.

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Figure 3: Plot of $u(x,t)$ when $t = \frac{3a}{2c}$.

**The String Profile for $t = \frac{2a}{c}$**

Setting $t = \frac{2a}{c}$, we find that $x \pm ct = x \pm 2a$. The first interval of $x$ we consider is the one where the left and right-most points, $x \pm 2a$, are outside the interval, $(-a,a)$, on the left side. The right-most point, $x + 2a$, touches the left end of the interval, $(-a,a)$, at the highest value of $x$.

For $x < -3a$, \((x - 2a, x + 2a) \cap (-a,a) = (-\infty, -a) \cap (-a,a) = \emptyset\)
\[\rightarrow \ u \left( x, \frac{2a}{c} \right) = \frac{1}{2c} \int_{-a}^{x} 1 \, ds = 0\]

The next interval of $x$ we consider is the one where the right-most point, $x + 2a$, is inside the interval, $(-a,a)$, but the left-most point, $x - 2a$, is not. The distance from $x - 2a$ to $x + 2a$ is $4a$, which is bigger than the interval, $(-a,a)$. This means the next interval of $x$ will be from where we left off, $-3a$, to $-a$, where the right-most point, $x + 2a$, touches the right end of the interval, $(-a,a)$.

For $-3a < x < -a$, \((x - 2a, x + 2a) \cap (-a,a) = (-a, x + 2a)\)
\[\rightarrow \ u \left( x, \frac{2a}{c} \right) = \frac{1}{2c} \int_{-a}^{x+2a} 1 \, ds = \frac{1}{2c} (x + 3a)\]
Since the interval \((-a, a)\) is only 2a units long, the next interval of \(x\) we consider is the one where the right-most point, \(x + 2a\), is outside the interval, \((-a, a)\), on the right, and the left-most point, \(x - 2a\), is still outside the left end. This interval of \(x\) will start from where we left off, \(-a\), and go to \(a\), where the left-most point touches the left end of the interval, \((-a, a)\).

For \(-a < x < a\), \((x - 2a, x + 2a) \cap (-a, a) = (-a, a)\)

\[
\rightarrow u\left(x, \frac{2a}{c}\right) = \frac{1}{2c} \int_{-a}^{a} 1 \, ds = \frac{a}{c}
\]

The next interval of \(x\) we consider is the one where the left-most point, \(x - 2a\), is inside the interval, \((-a, a)\), and the right-most point, \(x + \frac{3a}{2}\), is outside the interval, \((-a, a)\), on the right side. This interval of \(x\) will start from where we left off, \(a\), and go until \(3a\), where the left-most point touches the right end of \((-a, a)\).

For \(a < x < 3a\), \((x - 2a, x + 2a) \cap (-a, a) = (x - 2a, a)\)

\[
\rightarrow u\left(x, \frac{2a}{c}\right) = \frac{1}{2c} \int_{x-2a}^{a} 1 \, ds = \frac{1}{2c} (3a - x)
\]

The last interval of \(x\) we consider is the one where the left and right-most points, \(x \pm 2a\), are outside the interval, \((-a, a)\), on the right side. This interval of \(x\) will start from where we left off, \(3a\), and go to \(\infty\). The left-most point, \(x - 2a\), touches the right end of the interval, \((-a, a)\), at the lowest value of \(x\).

For \(x > 3a\), \((x - 2a, x + 2a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset\)

\[
\rightarrow u\left(x, \frac{2a}{c}\right) = \frac{1}{2c} \int_{a}^{\infty} 1 \, ds = 0
\]

Now that we know \(u\left(x, \frac{2a}{c}\right)\) for all \(x\), we can plot it.

![Figure 4](http://www.stemjock.com)
The String Profile for $t = \frac{5a}{c}$

Setting $t = \frac{5a}{c}$, we find that $x \pm ct = x \pm 5a$. The first interval of $x$ we consider is the one where the left and right-most points, $x \pm 5a$, are outside the interval, $(-a, a)$, on the left side. The right-most point, $x + 5a$, touches the left end of the interval, $(-a, a)$, at the highest value of $x$.

For $x < -6a$, $(x - 5a, x + 5a) \cap (-a, a) = (-\infty, -a) \cap (-a, a) = \emptyset$

\[ \Rightarrow u \left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{-\infty} 1 \, ds = 0 \]

The next interval of $x$ we consider is the one where the right-most point, $x + 5a$, is inside the interval, $(-a, a)$, but the left-most point, $x - 5a$, is not. The distance from $x - 5a$ to $x + 5a$ is $10a$, which is bigger than the interval, $(-a, a)$. This means the next interval of $x$ will be from where we left off, $-6a$, to $-4a$, where the left-most point, $x + 5a$, touches the right end of $(-a, a)$.

For $-6a < x < -4a$, $(x - 5a, x + 5a) \cap (-a, a) = (-a, x + 5a)$

\[ \Rightarrow u \left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{x + 5a} 1 \, ds = \frac{1}{2c} (x + 6a) \]

Since the interval $(-a, a)$ is only $2a$ units long, the next interval of $x$ we consider is the one where the right-most point, $x + 5a$, is outside the interval, $(-a, a)$, on the right and the left-most point, $x - 5a$, is still outside the left end. This interval of $x$ will start from where we left off, $-4a$, and go to $4a$, where the left-most point, $x - 5a$, touches the left end of the interval, $(-a, a)$.

For $-4a < x < 4a$, $(x - 5a, x + 5a) \cap (-a, a) = (-a, a)$

\[ \Rightarrow u \left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{-a}^{a} 1 \, ds = \frac{a}{c} \]

The next interval we consider is the one where the left-most point, $x - 5a$, is inside the interval, $(-a, a)$, and the right-most point, $x + 5a$, is outside $(-a, a)$ on the right side. The interval of $x$ will start from where we left off, $4a$, and go to $6a$, where the left-most point, $x - 5a$, touches the right end of the interval, $(-a, a)$.

For $4a < x < 6a$, $(x - 5a, x + 5a) \cap (-a, a) = (x - 5a, a)$

\[ \Rightarrow u \left(x, \frac{2a}{c}\right) = \frac{1}{2c} \int_{x - 5a}^{a} 1 \, ds = \frac{1}{2c} (6a - x) \]

The last interval of $x$ we consider is the one where the left and right-most points, $x \pm 5a$, are outside the interval, $(-a, a)$, on the right side. The interval of $x$ will start from where we left off, $6a$, and go to $\infty$. The left-most point, $x - 5a$, touches the right end of the interval, $(-a, a)$, at the lowest value of $x$.

For $x > 6a$, $(x - 5a, x + 5a) \cap (-a, a) = (a, \infty) \cap (-a, a) = \emptyset$

\[ \Rightarrow u \left(x, \frac{5a}{c}\right) = \frac{1}{2c} \int_{a}^{\infty} 1 \, ds = 0 \]

Now that we know $u \left(x, \frac{5a}{c}\right)$ for all $x$, we can plot it.

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Figure 5: Plot of $u(x,t)$ when $t = \frac{5a}{c}$. 