Exercise 3

Show that the wave equation has the following invariance properties.

(a) Any translate $u(x-y,t)$, where y is fixed, is also a solution.

(b) Any derivative, say u_x, of a solution is also a solution.

(c) The dilated function $u(ax,at)$ is also a solution, for any constant a.

Solution

Assume that $u = u(x,t)$ is a solution to the wave equation. Then $u_{tt} = c^2 u_{xx}$ is true.

Part (a)

Here we will show that $u = u(x-y,t)$, where y is a constant, satisfies the wave equation. Let $z = x - y$ so that $u = (z,t)$ and use the chain rule.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_z \cdot 1 + u_t \cdot 0 = u_z \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u_x}{\partial t}\frac{\partial t}{\partial x} = u_{zz} \cdot 1 + u_{zt} \cdot 0 = u_{zz} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial t} = u_z \cdot 0 + u_t \cdot 1 = u_t \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial u_t}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u_t}{\partial t}\frac{\partial t}{\partial t} = u_{tz} \cdot 0 + u_{tt} \cdot 1 = u_{tt}
\end{aligned}$$

Substituting u_{zz} for u_{xx} in the wave equation yields $u_{tt} = c^2 u_{zz}$. Therefore, any translate of a solution to the wave equation also satisfies it.

Part (b)

Here we will show that any derivative of u is a solution to the wave equation too. Let's start with u_x. Take two derivatives with respect to t and take two derivatives with respect to x.

$$\begin{cases} (u_x)_{tt} = u_{xtt} = u_{ttx} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 u_{xxx} \\ (u_x)_{xx} = u_{xxx} \end{cases}$$

Hence, $(u_x)_{tt} = c^2 (u_x)_{xx}$, and the derivative with respect to x of a solution to the wave equation also satisfies it. Now let's consider u_t. If we take the same derivatives, we get the following.

$$\begin{cases} (u_t)_{tt} = u_{ttt} \\ (u_t)_{xx} = u_{txx} = u_{xxt} = (u_{xx})_t = \left(\frac{1}{c^2} u_{tt}\right)_t = \frac{1}{c^2} u_{ttt} \end{cases}$$

Hence, $(u_t)_{tt} = c^2 (u_t)_{xx}$, and the derivative with respect to t of a solution to the wave equation also satisfies it.

Part (c)

Here we will show that the dilated function, $u = u(ax, at)$, is a solution to the wave equation as well. Let $r = ax$ and $s = at$ so that $u = u(r, s)$ and use the chain rule.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r \cdot a + u_s \cdot 0 = au_r \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial s}\frac{\partial s}{\partial x} = au_{rr} \cdot a + au_{rs} \cdot 0 = a^2 u_{rr} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = u_r \cdot 0 + u_s \cdot a = au_s \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial u_t}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u_t}{\partial s}\frac{\partial s}{\partial t} = au_{sr} \cdot 0 + au_{ss} \cdot a = a^2 u_{ss}
\end{aligned}$$

Substituting $a^2 u_{ss}$ for u_{tt} and $a^2 u_{rr}$ for u_{xx} in the wave equation yields $a^2 u_{ss} = c^2 a^2 u_{rr}$, which means $u_{ss} = c^2 u_{rr}$. Therefore, the dilated function also satisfies the wave equation.