Exercise 5

For the *damped* string, equation (1.3.3), show that the energy decreases.

Solution

The equation of motion for the damped string is given by equation (1.3.3) in the book.

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

The kinetic energy is defined to be

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2 \, dx.$$

If we take the derivative with respect to t, we get the following.

$$\frac{d(\text{KE})}{dt} = \frac{1}{2}\rho \frac{d}{dt}\int_{-\infty}^{\infty} u_t^2 \, dx$$
$$\frac{d(\text{KE})}{dt} = \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u_t^2) \, dx$$
$$\frac{d(\text{KE})}{dt} = \frac{1}{2}\rho \int_{-\infty}^{\infty} 2u_t u_{tt} \, dx$$

Upon substituting $u_{tt} = c^2 u_{xx} - r u_t$, we get

$$\frac{d(\text{KE})}{dt} = \frac{1}{2}\rho \int_{-\infty}^{\infty} 2u_t (c^2 u_{xx} - r u_t) \, dx$$
$$\frac{d(\text{KE})}{dt} = \rho c^2 \int_{-\infty}^{\infty} u_t u_{xx} \, dx - \rho r \int_{-\infty}^{\infty} u_t^2 \, dx.$$

Integrate the term on the left by parts. Also, note that $c^2 = T/\rho$ for a string.

$$\frac{d(\text{KE})}{dt} = \left(\underbrace{T u_t u_x \big|_{-\infty}^{\infty}}_{=0} - T \int_{-\infty}^{\infty} u_{tx} u_x \, dx \right) - \rho r \int_{-\infty}^{\infty} u_t^2 \, dx$$
$$\frac{d(\text{KE})}{dt} = -T \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}(u_x^2) \, dx - \rho r \int_{-\infty}^{\infty} u_t^2 \, dx$$
$$\frac{d(\text{KE})}{dt} = -\frac{1}{2}T \frac{d}{dt}\int_{-\infty}^{\infty} u_x^2 \, dx - \rho r \int_{-\infty}^{\infty} u_t^2 \, dx$$

Since potential energy is defined as

$$\text{PE} = \frac{1}{2}T \int_{-\infty}^{\infty} u_x^2 \, dx$$

and the total energy is $E = \text{KE} + \text{PE}$,

$$\frac{dE}{dt} = \frac{d(\text{KE})}{dt} + \frac{d(\text{PE})}{dt} = -\rho r \int_{-\infty}^{\infty} u_t^2 \, dx < 0.$$

Therefore, the energy decreases in time.