Exercise 2

For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.

(b) Show that both $e(x,t)$ and $p(x,t)$ also satisfy the wave equation.

Solution

Part (a)

The fact that $u(x,t)$ is a solution of the wave equation implies that $u_{tt} = u_{xx}$. From the definitions we have

$$\begin{cases} e = \dfrac{1}{2}(u_t^2 + u_x^2) \\ p = u_t u_x \end{cases}.$$

Taking the derivative of e with respect to t and the derivative of p with respect to x, we obtain

$$\begin{cases} \dfrac{\partial e}{\partial t} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial t}(u_t^2) + \dfrac{\partial}{\partial t}(u_x^2)\right] = \dfrac{\cancel{1}}{\cancel{2}}(\cancel{2}u_t u_{tt} + \cancel{2}u_x u_{xt}) \\ \dfrac{\partial p}{\partial x} = \dfrac{\partial}{\partial x}(u_t u_x) = u_{tx}u_x + u_t u_{xx} \end{cases}.$$

We can substitute u_{tt} for u_{xx} in the top equation and replace u_{tx} with u_{xt} in the bottom equation.

$$\begin{cases} \dfrac{\partial e}{\partial t} = u_t u_{xx} + u_x u_{xt} \\ \dfrac{\partial p}{\partial x} = u_x u_{xt} + u_t u_{xx} \end{cases}$$

Therefore,

$$\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}.$$

Taking the derivative of e with respect to x and the derivative of p with respect to t, we obtain

$$\begin{cases} \dfrac{\partial e}{\partial x} = \dfrac{1}{2}\left[\dfrac{\partial}{\partial x}(u_t^2) + \dfrac{\partial}{\partial x}(u_x^2)\right] = \dfrac{\cancel{1}}{\cancel{2}}(\cancel{2}u_t u_{tx} + \cancel{2}u_x u_{xx}) \\ \dfrac{\partial p}{\partial t} = \dfrac{\partial}{\partial t}(u_t u_x) = u_{tt}u_x + u_t u_{xt} \end{cases}.$$

We can substitute u_{xx} for u_{tt} and u_{tx} for u_{xt} in the top equation.

$$\begin{cases} \dfrac{\partial e}{\partial x} = u_t u_{xt} + u_x u_{tt} \\ \dfrac{\partial p}{\partial t} = u_x u_{tt} + u_t u_{xt} \end{cases}$$

Therefore,

$$\frac{\partial e}{\partial x} = \frac{\partial p}{\partial t}.$$

Part (b)

We established in part (a) that $p_t = e_x$ and $p_x = e_t$. That is,

$$\begin{cases} p_t = e_x \\ p_x = e_t \end{cases}.$$

Taking the derivative with respect to t on both sides of the top and the derivative with respect to x on both sides of the bottom yields

$$\begin{cases} p_{tt} = e_{xt} \\ p_{xx} = e_{tx} \end{cases}.$$

And since $e_{xt} = e_{tx}$, it means that $p_{tt} = p_{xx}$. On the other hand, taking the derivative with respect to x on both sides of the top and the derivative with respect to t on both sides of the bottom yields

$$\begin{cases} p_{tx} = e_{xx} \\ p_{xt} = e_{tt} \end{cases}.$$

And since $p_{tx} = p_{xt}$, it means that $e_{tt} = e_{xx}$. Therefore, $p(x,t)$ and $e(x,t)$ satisfy the wave equation.