Exercise 2

For a solution $u(x,t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.

(a) Show that $\partial e / \partial t = \partial p / \partial x$ and $\partial p / \partial t = \partial e / \partial x$.

(b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.

Solution

Part (a)

The fact that $u(x, t)$ is a solution of the wave equation implies that $u_{tt} = u_{xx}$. From the definitions we have

\[
\begin{cases}
  e = \frac{1}{2}(u_t^2 + u_x^2) \\
  p = u_t u_x
\end{cases}
\]

Taking the derivative of $e$ with respect to $t$ and the derivative of $p$ with respect to $x$, we obtain

\[
\begin{cases}
  \frac{\partial e}{\partial t} = \frac{1}{2} \left[ \frac{\partial}{\partial t} (u_t^2) + \frac{\partial}{\partial t} (u_x^2) \right] = \frac{1}{2} (2u_t u_{tt} + 2u_x u_{xt}) \\
  \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} (u_t u_x) = u_{tx} u_x + u_t u_{xx}
\end{cases}
\]

We can substitute $u_{tt}$ for $u_{xx}$ in the top equation and replace $u_{tx}$ with $u_{xt}$ in the bottom equation.

\[
\begin{cases}
  \frac{\partial e}{\partial t} = u_t u_{xx} + u_{tx} u_x \\
  \frac{\partial p}{\partial x} = u_{xt} u_x + u_t u_{xx}
\end{cases}
\]

Therefore, $\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$.

Taking the derivative of $e$ with respect to $x$ and the derivative of $p$ with respect to $t$, we obtain

\[
\begin{cases}
  \frac{\partial e}{\partial x} = \frac{1}{2} \left[ \frac{\partial}{\partial x} (u_t^2) + \frac{\partial}{\partial x} (u_x^2) \right] = \frac{1}{2} (2u_t u_{tx} + 2u_x u_{xx}) \\
  \frac{\partial p}{\partial t} = \frac{\partial}{\partial t} (u_t u_x) = u_{tt} u_x + u_t u_{xt}
\end{cases}
\]

We can substitute $u_{xx}$ for $u_{tt}$ and $u_{tx}$ for $u_{xt}$ in the top equation.

\[
\begin{cases}
  \frac{\partial e}{\partial x} = u_t u_{xt} + u_x u_{tt} \\
  \frac{\partial p}{\partial t} = u_{xt} u_x + u_t u_{xx}
\end{cases}
\]

Therefore, $\frac{\partial e}{\partial x} = \frac{\partial p}{\partial t}$.
Part (b)

We established in part (a) that \( p_t = e_x \) and \( p_x = e_t \). That is,

\[
\begin{aligned}
\begin{cases}
  p_t &= e_x \\
  p_x &= e_t 
\end{cases}
\end{aligned}
\]

Taking the derivative with respect to \( t \) on both sides of the top and the derivative with respect to \( x \) on both sides of the bottom yields

\[
\begin{aligned}
\begin{cases}
  p_{tt} &= e_{xt} \\
  p_{xx} &= e_{tx} 
\end{cases}
\end{aligned}
\]

And since \( e_{xt} = e_{tx} \), it means that \( p_{tt} = p_{xx} \). On the other hand, taking the derivative with respect to \( x \) on both sides of the top and the derivative with respect to \( t \) on both sides of the bottom yields

\[
\begin{aligned}
\begin{cases}
  p_{tx} &= e_{xx} \\
  p_{xt} &= e_{tt} 
\end{cases}
\end{aligned}
\]

And since \( p_{tx} = p_{xt} \), it means that \( e_{tt} = e_{xx} \). Therefore, \( p(x,t) \) and \( e(x,t) \) satisfy the wave equation.