Exercise 3

Consider the diffusion equation $u_t = u_{xx}$ in the interval (0,1) with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x,t) > 0$ at all interior points $0 < x < 1$, $0 < t < \infty$.

(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x,t)$ over $0 \leq x \leq 1$. Show that $\mu(t)$ is a decreasing (i.e. nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)

(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

Solution

Part (a)

According to the minimum principle, the lowest value of u occurs on the boundary or initially. Since $0 < x < 1$ and the initial condition is $u(x,0) = 1 - x^2$, $u > 0$ on the interior points of the bar initially. Now consider u on the boundary. Because the ends of the rod are held at zero temperature for all time, i.e. $u(0,t) = u(1,t) = 0$, $u = 0$ is the minimum value. This means that the initial profile will drop down to zero as time increases. As long as the time remains finite, though, u will never actually reach 0. Therefore, $u(x,t) > 0$ at all interior points $(0 < x < 1)$ for $0 < t < \infty$.

Part (b)

The goal here is to show that the maximum of u, $\mu(t)$, is a decreasing function of time. That is,

$$\frac{d\mu}{dt} < 0$$

for all $t > 0$. Following the hint, suppose the maximum occurs at the x-coordinate, $X(t)$. Then

$$\mu(t) = u\left(x = X(t), t\right).$$

Take the derivative of this with respect to t. We have to use the chain rule since both arguments are functions of t.

$$\frac{d\mu}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = u_x \frac{dX}{dt} + u_t$$

Now we integrate both sides over the length of the bar.

$$\int_0^1 \frac{d\mu}{dt}\,dx = \int_0^1 \left(u_x \frac{dX}{dt} + u_t\right) dx$$

$$\frac{d\mu}{dt}\int_0^1 dx = \frac{dX}{dt}\int_0^1 u_x\,dx + \int_0^1 u_t\,dx$$

$$\frac{d\mu}{dt} = \frac{dX}{dt}\Bigg[\underbrace{u(1,t)}_{=0} - \underbrace{u(0,t)}_{=0}\Bigg] + \int_0^1 \frac{\partial u}{\partial t}\,dx$$

$$\frac{d\mu}{dt} = \frac{d}{dt}\int_0^1 u\,dx$$

The integral of u over the length of the bar represents the area under the curve. Because the concentration goes from $u = 1 - x^2$ to $u = 0$, the area under u gets smaller and smaller over time. Hence, the right side is less than zero. Therefore,

$$\frac{d\mu}{dt} < 0,$$

and this means $\mu(t)$ is a decreasing function of t.

Part (c)

The solution to the diffusion equation that satisfies the given boundary conditions and initial condition is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where

$$A_n = \frac{2}{n^3\pi^3}[2 - 2(-1)^n + n^2\pi^2].$$

Shown below are graphs of u as a function of x for five different times.

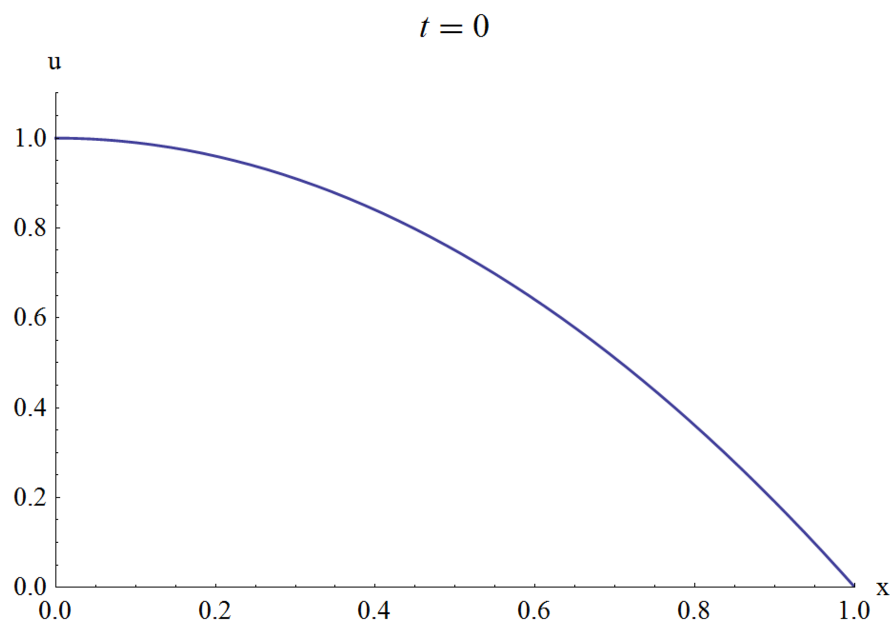

Figure 1: The concentration profile at $t = 0$.

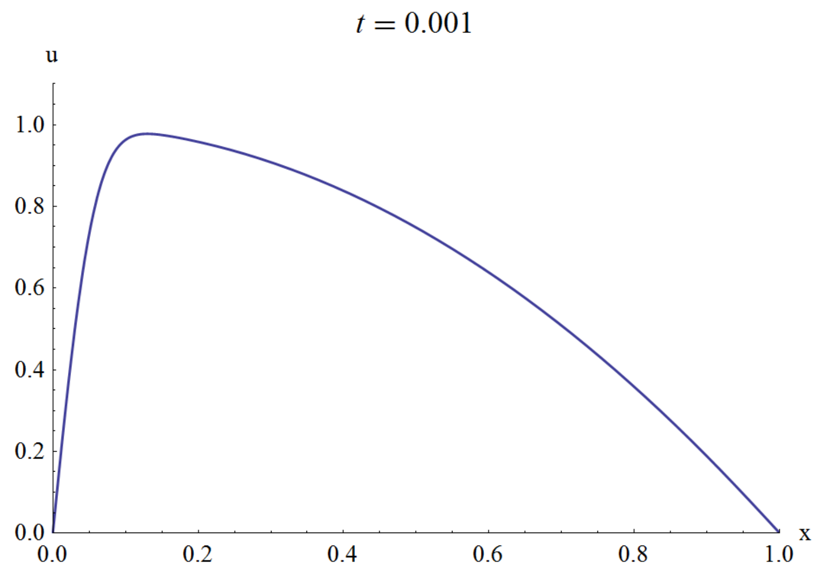

Figure 2: The concentration profile at $t = 0.001$.

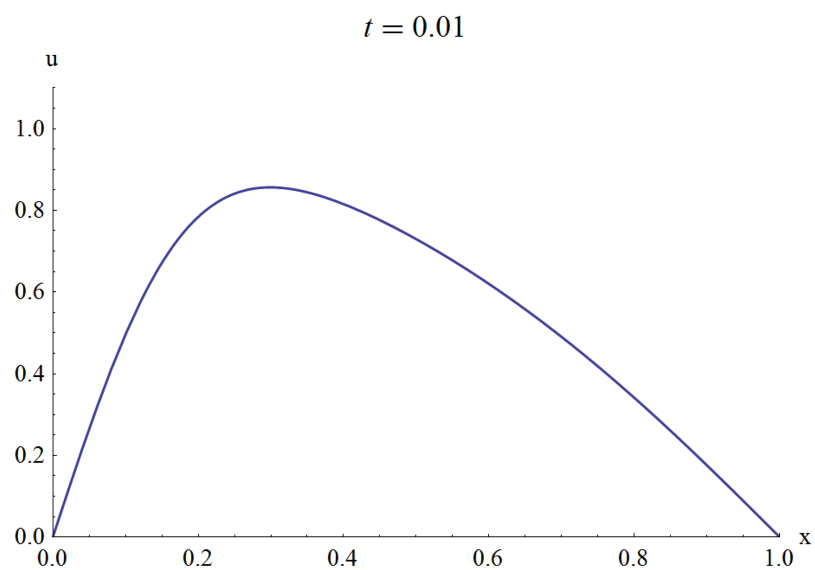

Figure 3: The concentration profile at $t = 0.01$.

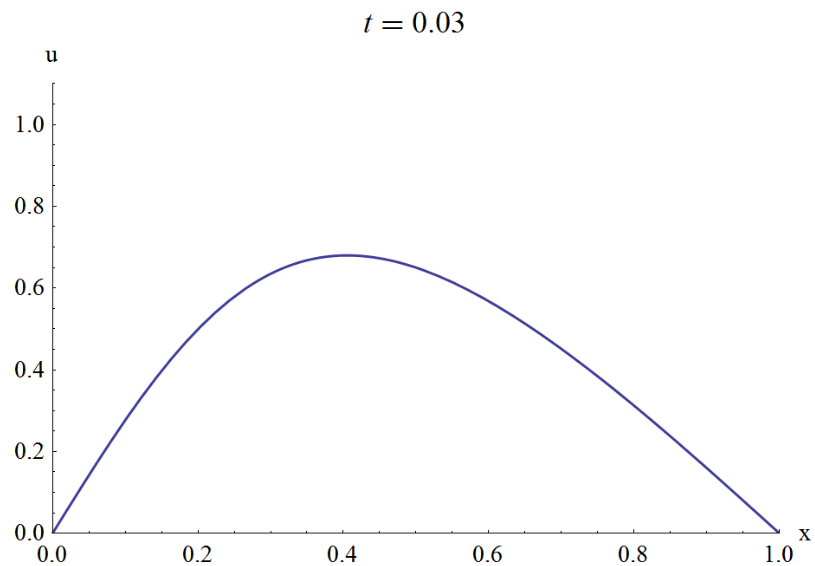

Figure 4: The concentration profile at $t = 0.03$.

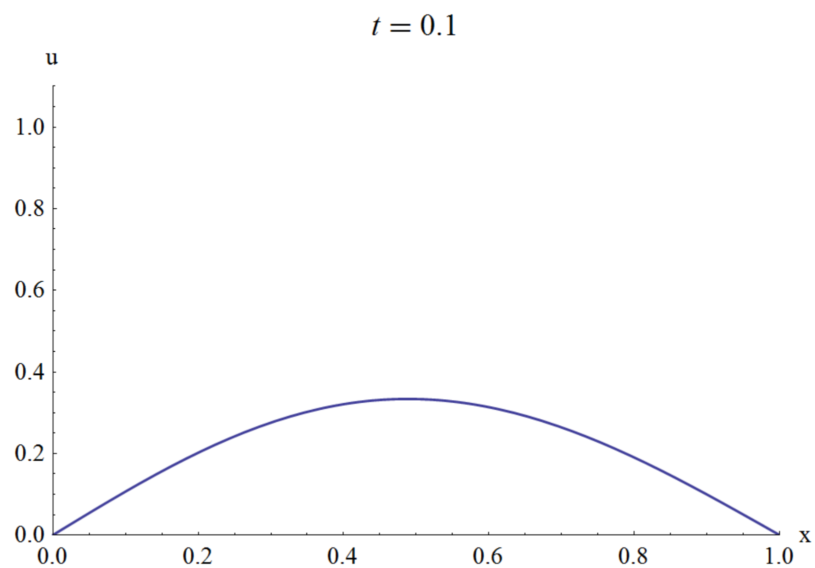

Figure 5: The concentration profile at $t = 0.1$.