Exercise 3

Consider the diffusion equation $u_t = u_{xx}$ in the interval (0,1) with $u(0,t) = u(1,t) = 0$ and $u(x,0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x,t) > 0$ at all interior points $0 < x < 1$, $0 < t < \infty$.

(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x,t)$ over $0 \leq x \leq 1$. Show that $\mu(t)$ is a decreasing (i.e. nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t),t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)

(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)